Edwin H. Spanier

Algebraic Topology

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Algebraic Topology

PREFACE TO THE SECOND SPRINGER PRINTING

IN THE MORE THAN TWENTY YEARS SINCE THE FIRST APPEARANCE OF *Algebraic Topology* the book has met with favorable response both in its use as a text and as a reference. It was the first comprehensive treatment of the fundamentals of the subject. Its continuing acceptance attests to the fact that its content and organization are still as timely as when it first appeared. Accordingly it has not been revised.

Many of the proofs and concepts first presented in the book have become standard and are routinely incorporated in newer books on the subject. Despite this, *Algebraic Topology* remains the best complete source for the material which every young algebraic topologist should know. Springer-Verlag is to be commended for its willingness to keep the book in print for future topologists.

For the current printing all of the misprints known to me have been corrected and the bibliography has been updated.

Berkeley, California
December 1989

Edwin H. Spanier

PREFACE

THIS BOOK IS AN EXPOSITION OF THE FUNDAMENTAL IDEAS OF ALGEBRAIC topology. It is intended to be used both as a text and as a reference. Particular emphasis has been placed on naturality, and the book might well have been titled *Functorial Topology.* The reader is not assumed to have prior knowledge of algebraic topology, but he is assumed to know something of general topology and algebra and to be mathematically sophisticated. Specific prerequisite material is briefly summarized in the Introduction.

Since *Algebraic Topology* is a text, the exposition in the earlier chapters is a good deal slower than in the later chapters. The reader is expected to develop facility for the subject as he progresses, and accordingly, the further he is in the book, the more he is called upon to fill in details of proofs. Because it is also intended as a reference, some attempt has been made to include basic concepts whether they are used in the book or not. As a result, there is more material than is usually given in courses on the subject.

The material is organized into three main parts, each part being made up of three chapters. Each chapter is broken into several sections which treat individual topics with some degree of thoroughness and are the basic organizational units of the text. In the first three chapters the underlying theme is the fundamental group. This is defined in Chapter One, applied in Chapter Two in the study of covering spaces, and described by means of generators and relations in Chapter Three, where polyhedra are introduced. The concept of functor and its applicability to topology are stressed here to motivate interest in the other functors of algebraic topology.

Chapters Four, Five, and Six are devoted to homology theory. Chapter Four contains the first definitions of homology, Chapter Five contains further algebraic concepts such as cohomology, cup products, and cohomology operations, and Chapter Six contains a study of topological manifolds. With each new concept introduced applications are presented to illustrate its utility.

The last three chapters study homotopy theory. Basic facts about homotopy groups are considered in Chapter Seven, applications to obstruction theory are presented in Chapter Eight, and some computations of homotopy groups of spheres are given in Chapter Nine. Main emphasis is on the application to geometry of the algebraic tools introduced earlier.

There is probably more material than can be covered in a year course. The core of a first course in algebraic topology is Chapter Four. This contains elementary facts about homology theory and some of its most important applications. A satisfactory one-semester first course for graduate students can be based on the first four chapters, either omitting or treating briefly Secs. 5 and 6 of Chapter One, Secs. 7 and 8 of Chapter Two, Sec. 8 of Chapter Three, and Sec. 8 of Chapter Four. A second one-semester course can be based on Chapters Five, Six, Seven, and Eight or on Chapters Five, Seven, Eight, and Nine. For students with knowledge of homology theory and related algebraic concepts a course in homotopy theory based on the last three chapters is quite feasible.

Each chapter is followed by a collection of exercises. These are grouped into sets, each set being devoted to a single topic or a few related topics. With few exceptions, none of the exercises is referred to in the body of the text or in the sequel. There are various types of exercises. Some are examples of the general theory developed in the preceding chapter, some treat special cases of general topics discussed later, and some are devoted to topics not discussed in the text at all. There are routine exercises as well as more difficult ones, the latter frequently with hints of how to attack them. Occasionally a topic related to material in the text is developed in a set of exercises devoted to it.

Examples in the text are usually presented with little or no indication of why they have the stated properties. This is true both of examples illustrating new concepts and of counterexamples. The verification that an example has the desired properties is left to the reader as an exercise.

The symbol ■ is used to denote the end of a proof. It is also used at the end of a statement whose proof has been given before the statement or which follows easily from previous results. Bibliographical references are by footnotes in the text. Items in each section and in each exercise set are numbered consecutively in a single list. References to items in a different section are by triples indicating, respectively, the chapter, the section or exercise set, and the number of the item in the section. Thus 3.2.2 is item 2 in Sec. 2 of Chapter Three (and 3.2 of the Introduction is item 2 in Sec. 3 of the Introduction).

The idea of writing this book originated with the existence of lecture notes based on two courses I gave at the University of Chicago in 1955. It is a pleasure to acknowledge here my indebtedness to the authors of those notes, Guido Weiss for notes of the first course, and Edward Halpern for notes of the second course. In the years since then, the subject has changed substantially and my plans for the book changed along with it, so that the present volume differs in many ways from the original notes.

The final manuscript and galley proofs were read by Per Holm. He made a number of useful suggestions which led to improvements in the text. For his comments and for his friendly encouragement at dark moments, I am sincerely grateful to him. The final manuscript was typed by Mrs. Ann Harrington and Mrs. Ollie Cullers, to both of whom I express my thanks for their patience and cooperation.

I thank the Air Force Office of Scientific Research for a grant enabling me to devote all my time during the academic year 1962–63 to work on this book. I also thank the National Science Foundation for supporting, over a period of years, my research activities some of which are discussed here.

Edwin H. Spanier

LIST OF SYMBOLS

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ALGEBRAIC TOPOLOGY

INTRODUCTION

THE READER OF THIS BOOK IS ASSUMED TO HAVE A GRASP OF THE ELEMENTARY concepts of set theory, general topology, and algebra. Following are brief summaries of some concepts and results in these areas which are used in this book. Those listed explicitly are done so either because they may not be exactly standard or because they are of particular importance in the subsequent text.

1 SET THEORY[1]

The terms "set," "family," and "collection" are synonyms, and the term "class" is reserved for an aggregate which is not assumed to be a set (for example, the class of all sets). If X is a set and $P(x)$ is a statement which is either true or false for each element $x \in X$, then

[1] As a general reference see P. R. Halmos, *Naïve Set Theory*, D. Van Nostrand Company, Inc., Princeton, N.J., 1960.

$$\{x \in X \mid P(x)\}$$

denotes the subset of X for which $P(x)$ is true.

If $J = \{j\}$ is a set and $\{A_j\}$ is a family of sets indexed by J, their *union* is denoted by $\bigcup A_j$ (or by $\bigcup_{j \in J} A_j$), their *intersection* is denoted by $\bigcap A_j$ (or by $\bigcap_{j \in J} A_j$), their *cartesian product* is denoted by $\times A_j$ (or by $\times_{j \in J} A_j$), and their *set sum* (sometimes called their *disjoint union*) is denoted by $\bigvee A_j$ (or by $\bigvee_{j \in J} A_j$) and is defined by $\bigvee A_j = \bigcup (j \times A_j)$. In case $J = \{1,2, \ldots ,n\}$, we also use the notation $A_1 \cup A_2 \cup \cdots \cup A_n$, $A_1 \cap A_2 \cap \cdots \cap A_n$, $A_1 \times A_2 \times \cdots \times A_n$, and $A_1 \vee A_2 \vee \cdots \vee A_n$, respectively, for the union, intersection, cartesian product, and set sum.

A *function* (or *map*) f from A to B is denoted by $f\colon A \to B$. The set of all functions from A to B is denoted by B^A. If $A' \subset A$, there is an *inclusion map* $i\colon A' \to A$, and we use the notation $i\colon A' \subset A$ to indicate that A' is a subset of A and i is the inclusion map. The inclusion map from a set A to itself is called the *identity map of* A and is denoted by 1_A. If $J' \subset J$, there is an inclusion map

$$i_{J'}\colon \bigvee_{j \in J'} A_j \subset \bigvee_{j \in J} A_j$$

An *equivalence relation* in a set A is a relation $\sim$ between elements of A which is *reflexive* (that is, $a \sim a$ for all $a \in A$), *symmetric* (that is, $a \sim a'$ implies $a' \sim a$ for $a, a' \in A$), and *transitive* (that is, $a \sim a'$ and $a' \sim a''$ imply $a \sim a''$ for $a, a', a'' \in A$). The *equivalence class* of $a \in A$ with respect to $\sim$ is the subset $\{a' \in A \mid a \sim a'\}$. The set of all equivalence classes of elements of A with respect to $\sim$ is denoted by $A/\sim$ and is called a *quotient set of* A. There is a *projection map* $A \to A/\sim$ which sends $a \in A$ to its equivalence class. If J' is a nonempty subset of J, there is also a *projection map*

$$p_{J'}\colon \times_{j \in J} A_j \to \times_{j \in J'} A_j$$

(which is a projection map in the sense above).

Given functions $f\colon A \to B$ and $g\colon B \to C$, their *composite* $g \circ f$ (also denoted by gf) is the function from A to C defined by $(g \circ f)(a) = g(f(a))$ for $a \in A$. If $A' \subset A$ and $f\colon A \to B$, the *restriction of f to A'* is the function $f \mid A'\colon A' \to B$ defined by $(f \mid A')(a') = f(a')$ for $a' \in A'$ (thus $f \mid A' = f \circ i$, where $i\colon A' \subset A$), and the function f is called an *extension of $f \mid A'$ to A*.

An *injection* (or *injective function*) is a function $f\colon A \to B$ such that $f(a_1) = f(a_2)$ implies $a_1 = a_2$ for $a_1, a_2 \in A$. A *surjection* (or *surjective function*) is a function $f\colon A \to B$ such that $b \in B$ implies that there is $a \in A$ with $f(a) = b$. A *bijection* (also called a *bijective function* or a *one-to-one correspondence*) is a function which is both injective and surjective.

A *partial order* in a set A is a relation $\leq$ between elements of A which is reflexive and transitive (note that it is not assumed that $a \leq a'$ and $a' \leq a$ imply $a = a'$). A *total order* (or *simple order*) in A is a partial order in A such that for $a, a' \in A$ either $a \leq a'$ or $a' \leq a$ and which is *antisymmetric* (that is, $a \leq a'$ and $a' \leq a$ imply $a = a'$). A *partially ordered set* is a set with a partial order, and a *totally ordered set* is a set with a total order.

1 ZORN'S LEMMA *A partially ordered set in which every simply ordered subset has an upper bound contains maximal elements.*

A *directed set* Λ is a set with a partial-order relation $\le$ such that for $\alpha, \beta \in \Lambda$ there is $\gamma \in \Lambda$ with $\alpha \le \gamma$ and $\beta \le \gamma$. A *direct system of sets* $\{A^\alpha, f_\alpha{}^\beta\}$ consists of a collection of sets $\{A^\alpha\}$ indexed by a directed set $\Lambda = \{\alpha\}$ and a collection of functions $f_\alpha{}^\beta\colon A^\alpha \to A^\beta$ for every pair $\alpha \le \beta$ such that

(*a*) $f_\alpha{}^\alpha = 1_{A^\alpha}\colon A^\alpha \subset A^\alpha$ for all $\alpha \in \Lambda$
(*b*) $f_\alpha{}^\gamma = f_\beta{}^\gamma \circ f_\alpha{}^\beta\colon A^\alpha \to A^\gamma$ for $\alpha \le \beta \le \gamma$ in Λ

The *direct limit* of the direct system, denoted by $\lim_{\to} \{A^\alpha\}$, is the set of equivalence classes of $\bigvee A^\alpha$ with respect to the equivalence relation $a^\alpha \sim a^\beta$ if there is γ with $\alpha \le \gamma$ and $\beta \le \gamma$ such that $f_\alpha{}^\gamma a^\alpha = f_\beta{}^\gamma a^\beta$. For each α there is a map $i_\alpha\colon A^\alpha \to \lim_{\to} \{A^\alpha\}$, and if $\alpha \le \beta$, then $i_\alpha = i_\beta \circ f_\alpha{}^\beta$.

2 *Given a direct system of sets $\{A^\alpha, f_\alpha{}^\beta\}$ and given a set B and for every $\alpha \in \Lambda$ a function $g_\alpha\colon A^\alpha \to B$ such that $g_\alpha = g_\beta \circ f_\alpha{}^\beta$ if $\alpha \le \beta$, there is a unique map $g\colon \lim_{\to} \{A^\alpha\} \to B$ such that $g \circ i_\alpha = g_\alpha$ for all $\alpha \in \Lambda$.*

3 *With the same notation as in theorem 2, the map g is a bijection if and only if both the following hold:*

(*a*) $B = \bigcup g_\alpha(A^\alpha)$
(*b*) $g_\alpha(a^\alpha) = g_\beta(a^\beta)$ *if and only if there is γ with $\alpha \le \gamma$ and $\beta \le \gamma$ such that $f_\alpha{}^\gamma(a^\alpha) = f_\beta{}^\gamma(a^\beta)$*

Let $\{A_j\}$ be a collection of sets indexed by $J = \{j\}$. Let Λ be the collection of finite subsets of J and define $\alpha \le \beta$ for $\alpha, \beta \in \Lambda$ if $\alpha \subset \beta$. Then Λ is a directed set and there is a direct system $\{A^\alpha\}$ defined by $A^\alpha = \bigvee_{j \in \alpha} A_j$, and if $\alpha \le \beta$, then $f_\alpha{}^\beta\colon A^\alpha \to A^\beta$ is the injection map. Let $g_\alpha\colon A^\alpha \to \bigvee_{j \in J} A_j$ be the injection map.

4 *With the above notation, there is a bijection $g\colon \lim_{\to} \{A^\alpha\} \to \bigvee_{j \in J} A_j$ such that $g \circ i_\alpha = g_\alpha$ (that is, any set sum is the direct limit of its finite partial set sums).*

An *inverse system of sets* $\{A_\alpha, f_\alpha{}^\beta\}$ consists of a collection of sets $\{A_\alpha\}$ indexed by a directed set $\Lambda = \{\alpha\}$ and a collection of functions $f_\alpha{}^\beta\colon A_\beta \to A_\alpha$ for $\alpha \le \beta$ such that

(*a*) $f_\alpha{}^\alpha = 1_{A_\alpha}\colon A_\alpha \subset A_\alpha$ for $\alpha \in \Lambda$
(*b*) $f_\alpha{}^\gamma = f_\alpha{}^\beta \circ f_\beta{}^\gamma\colon A_\gamma \to A_\alpha$ for $\alpha \le \beta \le \gamma$ in Λ

The *inverse limit* of the inverse system, denoted by $\lim_{\leftarrow} \{A_\alpha\}$, is the subset of $\times A_\alpha$ consisting of all points (a_α) such that if $\alpha \le \beta$, then $a_\alpha = f_\alpha{}^\beta a_\beta$. For each α there is a map $p_\alpha\colon \lim_{\leftarrow} \{A_\alpha\} \to A_\alpha$, and if $\alpha \le \beta$, then $p_\alpha = f_\alpha{}^\beta \circ p_\beta$.

5 *Given an inverse system of sets $\{A_\alpha, f_\alpha{}^\beta\}$ and given a set B and for every $\alpha \in \Lambda$ a function $g_\alpha\colon B \to A_\alpha$ such that $g_\alpha = f_\alpha{}^\beta \circ g_\beta$ if $\alpha \le \beta$, there is a unique function $g\colon B \to \lim_{\leftarrow} \{A_\alpha\}$ such that $g_\alpha = p_\alpha \circ g$ for all $\alpha \in \Lambda$.*

6 *With the same notation as in theorem 5, the map g is a bijection if and only if both the following hold:*

(*a*) $g_\alpha(b) = g_\alpha(b')$ *for all* $\alpha \in \Lambda$ *implies* $b = b'$
(*b*) *Given* $(a_\alpha) \in \times A_\alpha$ *such that* $a_\alpha = f_\alpha{}^\beta a_\beta$ *if* $\alpha \le \beta$, *there is* $b \in B$ *such that* $g_\alpha(b) = a_\alpha$ *for all* $\alpha \in \Lambda$

Let $\{A^j\}$ be a collection of sets indexed by $J = \{j\}$. Let Λ be the collection of finite nonempty subsets of J, and define $\alpha \le \beta$ for $\alpha, \beta \in \Lambda$ if $\alpha \subset \beta$. Then Λ is a directed set and there is an inverse system $\{A_\alpha\}$ defined by $A_\alpha = \times_{j \in \alpha} A^j$, and if $\alpha \le \beta$, $f_\alpha{}^\beta\colon A_\beta \to A_\alpha$ is the projection map. For each $\alpha \in \Lambda$ let $g_\alpha\colon \times_{j \in J} A^j \to A_\alpha$ be the projection map.

7 *With the above notation, there is a bijection* $g\colon \times_{j \in J} A^j \to \lim_{\leftarrow} \{A_\alpha\}$ *such that* $g_\alpha = p_\alpha \circ g$ *(that is, any cartesian product is the inverse limit of its finite partial cartesian products).*

2 GENERAL TOPOLOGY[1]

A topological space, also called a space, is not assumed to satisfy any separation axioms unless explicitly stated. Paracompact, normal, and regular spaces will always be assumed to be Hausdorff spaces. A continuous map from one topological space to another will also be called simply a map.

Given a set X and an indexed collection of topological spaces $\{X_j\}_{j \in J}$ and functions $f_j\colon X \to X_j$, the *topology induced on* X by the functions $\{f_j\}$ is the smallest or coarsest topology such that each f_j is continuous.

1 *The topology induced on X by functions* $\{f_j\colon X \to X_j\}$ *is characterized by the property that if Y is a topological space, a function* $g\colon Y \to X$ *is continuous if and only if* $f_j \circ g\colon Y \to X_j$ *is continuous for each* $j \in J$.

A *subspace* of a topological space X is a subset A of X topologized by the topology induced by the inclusion map $A \subset X$. A *discrete subset* of a topological space X is a subset such that every subset of it is closed in X. The *topological product* of an indexed collection of topological spaces $\{X_j\}_{j \in J}$ is the cartesian product $\times X_j$, given the topology induced by the projection maps $p_j\colon \times X_j \to X_j$ for $j \in J$. If $\{X_\alpha\}_{\alpha \in \Lambda}$ is an inverse system of topological spaces (that is, X_α is a topological space for $\alpha \in \Lambda$ and $f_\alpha{}^\beta\colon X_\beta \to X_\alpha$ is continuous for $\alpha \le \beta$) their inverse limit $\lim_{\leftarrow} \{X_\alpha\}$ is given the topology induced by the functions $p_\alpha\colon \lim_{\leftarrow} \{X_\alpha\} \to X_\alpha$ for $\alpha \in \Lambda$.

Given a set X and an indexed collection of topological spaces $\{X_j\}_{j \in J}$ and functions $g_j\colon X_j \to X$, the *topology coinduced on* X by the functions $\{g_j\}$ is the largest or finest topology such that each g_j is continuous.

[1] As general references see J. L. Kelley, *General Topology*, D. Van Nostrand Company, Inc., Princeton, N.J., 1955, and S. T. Hu, *Elements of General Topology*, Holden-Day, Inc., San Francisco, 1964.

2 *The topology coinduced on X by functions $\{g_j\colon X_j \to X\}$ is characterized by the property that if Y is any topological space, a function $f\colon X \to Y$ is continuous if and only if $f \circ g_j\colon X_j \to Y$ is continuous for each $j \in J$.*

A *quotient space* of a topological space X is a quotient set X' of X topologized by the topology coinduced by the projection map $X \to X'$. If $A \subset X$, then X/A will denote the quotient space of X obtained by identifying all of A to a single point. The *topological sum* of an indexed collection of topological spaces $\{X_j\}_{j \in J}$ is the set sum $\vee X_j$, given the topology coinduced by the injection maps $i_j\colon X_j \to \vee X_j$ for $j \in J$. If $\{X^\alpha\}_{\alpha \in \Lambda}$ is a direct system of topological spaces (that is, X^α is a topological space for $\alpha \in \Lambda$ and $f_\alpha{}^\beta\colon X^\alpha \to X^\beta$ is continuous for $\alpha \le \beta$) their direct limit $\lim_\to \{X^\alpha\}$ is given the topology coinduced by the functions $i_\alpha\colon X^\alpha \to \lim_\to \{X^\alpha\}$ for $\alpha \in \Lambda$.

Let $\mathcal{A} = \{A\}$ be a collection of subsets of a topological space X. X is said to have a *topology coherent with* $\mathcal{A}$ if the topology on X is coinduced from the subspaces $\{A\}$ by the inclusion maps $A \subset X$. (In the literature this topology is often called the *weak topology* with respect to $\mathcal{A}$.)

3 *A necessary and sufficient condition that X have a topology coherent with $\mathcal{A}$ is that a subset B of X be closed (or open) in X if and only if $B \cap A$ is closed (or open) in the subspace A for every $A \in \mathcal{A}$.*

4 *If $\mathcal{A}$ is an arbitrary open covering or a locally finite closed covering of X, then X has a topology coherent with $\mathcal{A}$.*

5 *Let X be a set and let $\{A_j\}$ be an indexed collection of topological spaces each contained in X and such that for each j and j', $A_j \cap A_{j'}$ is a closed (or open) subset of A_j and of $A_{j'}$ and the topology induced on $A_j \cap A_{j'}$ from A_j equals the topology induced on $A_j \cap A_{j'}$ from $A_{j'}$. Then the topology coinduced on X by the collection of inclusion maps $\{A_j \subset X\}$ is characterized by the properties that A_j is a closed (or open) subspace of X for each j and X has a topology coherent with the collection $\{A_j\}$.*

The topology on X in theorem 5 will be called *the topology coherent with* $\{A_j\}$. A *compactly generated space* is a Hausdorff space having a topology coherent with the collection of its compact subsets (this is the same as what is sometimes referred to as a Hausdorff *k-space*).

6 *A Hausdorff space which is either locally compact or satisfies the first axiom of countability is compactly generated.*

7 *If X is compactly generated and Y is a locally compact Hausdorff space, $X \times Y$ is compactly generated.*

If X and Y are topological spaces and $A \subset X$ and $B \subset Y$, then $\langle A;B\rangle$ denotes the set of continuous functions $f\colon X \to Y$ such that $f(A) \subset B$. Y^X denotes the space of continuous functions from X to Y, given the *compact-open topology* (which is the topology generated by the subbase $\{\langle K;U\rangle\}$, where K is a compact subset of X and U is an open subset of Y). If $A \subset X$

and $B \subset Y$, we use $(Y,B)^{(X,A)}$ to denote the subspace of Y^X of continuous functions $f\colon X \to Y$ such that $f(A) \subset B$. Let $E\colon Y^X \times X \to Y$ be the *evaluation map* defined by $E(f,x) = f(x)$. Given a function $g\colon Z \to Y^X$, the composite

$$Z \times X \xrightarrow{g \times 1} Y^X \times X \xrightarrow{E} Y$$

is a function from $Z \times X$ to Y.

8 THEOREM OF EXPONENTIAL CORRESPONDENCE *If X is a locally compact Hausdorff space and Y and Z are topological spaces, a map $g\colon Z \to Y^X$ is continuous if and only if $E \circ (g \times 1)\colon Z \times X \to Y$ is continuous.*

9 EXPONENTIAL LAW *If X is a locally compact Hausdorff space, Z is a Hausdorff space, and Y is a topological space, the function $\psi\colon (Y^X)^Z \to Y^{Z \times X}$ defined by $\psi(g) = E \circ (g \times 1)$ is a homeomorphism.*

10 *If X is a compact Hausdorff space and Y is metrized by a metric d, then Y^X is metrized by the metric d' defined by*

$$d'(f,g) = \sup \{d(f(x),g(x)) \mid x \in X\}$$

3 GROUP THEORY[1]

A homomorphism is called a *monomorphism, epimorphism, isomorphism,* respectively, if it is injective, surjective, bijective. If $\{G_j\}_{j \in J}$ is an indexed collection of groups, their *direct product* is the group structure on the cartesian product $\times\, G_j$ defined by $(g_j)(g_j') = (g_jg_j')$. If $\{G_\alpha\}$ is an inverse system of groups (that is, G_α is a group for each α and $f_\alpha{}^\beta\colon G_\beta \to G_\alpha$ is a homomorphism for $\alpha \leq \beta$), their inverse limit $\lim_{\leftarrow} \{G_\alpha\}$ (which is a set) is a subgroup of $\times\, G_\alpha$.

Let A be a subset of a group G. G is said to be *freely generated* by A and A is said to be a *free generating set* or *free basis* for G if, given any function $f\colon A \to H$, where H is a group, there exists a unique homomorphism $\varphi\colon G \to H$ which is an extension of f. A group is said to be *free* if it is freely generated by some subset. For any set A a *free group generated by* A is a group $F(A)$ containing A as a free generating set. Such groups $F(A)$ exist, and any two are canonically isomorphic.

1 *Any group is isomorphic to a quotient group of a free group.*

A *presentation* of a group G consists of a set A of *generators,* a set $B \subset F(A)$ of *relations,* and a function $f\colon A \to G$ such that the extension of f to a homomorphism $\varphi\colon F(A) \to G$ is an epimorphism whose kernel is the nor-

[1] As a general reference for elementary group theory see G. Birkhoff and S. MacLane, *A Survey of Modern Algebra,* The Macmillan Company, New York, 1953. For a discussion of free groups see R. H. Crowell and R. H. Fox, *Introduction to Knot Theory,* Ginn and Company, Boston, 1963.

mal subgroup of $F(A)$ generated by B. If A and B are both finite sets, the presentation is said to be *finite* and G is said to be *finitely presented.*

4 MODULES[1]

We are mainly interested in R modules where R is a principal ideal domain. However, we shall begin with some properties of R modules where R is a commutative ring with a unit which acts as the identity on every module. If $\varphi: A \to B$ is a homomorphism of R modules, then we have R modules

$$\begin{aligned} \ker \varphi &= \{a \in A \mid \varphi(a) = 0\} \subset A \\ \operatorname{im} \varphi &= \{b \in B \mid b = \varphi(a) \text{ for some } a \in A\} \subset B \\ \operatorname{coker} \varphi &= B/\operatorname{im} \varphi \end{aligned}$$

1 NOETHER ISOMORPHISM THEOREM *Let A and B be submodules of a module C and let $A + B$ be the submodule of C generated by $A \cup B$. The inclusion map $A \subset A + B$ sends $A \cap B$ into B and induces an isomorphism of $A/(A \cap B)$ with $(A + B)/B$.*

If $\{A_j\}_{j \in J}$ is an indexed collection of R modules, their direct product $\times A_j$ is an R module and their *direct sum* $\bigoplus A_j$ is an R module ($\bigoplus A_j$ is the submodule of $\times A_j$ consisting of those elements having only a finite number of nonzero coordinates). The inverse limit $\lim_{\leftarrow} \{A_\alpha\}$ of an inverse system of R modules (and homomorphisms $f_\alpha{}^\beta: A_\beta \to A_\alpha$ for $\alpha \le \beta$) is an R module, and the direct limit of a direct system of R modules (and homomorphisms) is an R module.

2 *Any R module is isomorphic to the direct limit of its finitely generated submodules directed by inclusion.*

If A and B are R modules, their *tensor product* $A \otimes B$ (also written $A \underset{R}{\otimes} B$) is an R module. For $a \in A$ and $b \in B$, there is a corresponding element $a \otimes b \in A \otimes B$. $A \otimes B$ is generated by the elements $\{a \otimes b \mid a \in A, b \in B\}$ with the relations (for $a, a' \in A$, $b, b' \in B$, and $r, r' \in R$)

$$\begin{aligned} (ra + r'a') \otimes b &= r(a \otimes b) + r'(a' \otimes b) \\ a \otimes (rb + r'b') &= r(a \otimes b) + r'(a \otimes b') \end{aligned}$$

In case A or B is also an R' module, then so is $A \underset{R}{\otimes} B$.

3 *For any R module A the homomorphisms $a \to a \otimes 1$ and $a \to 1 \otimes a$ define isomorphisms of A with $A \otimes R$ and $R \otimes A$.*

[1] As general references see H. Cartan and S. Eilenberg, *Homological Algebra,* Princeton University Press, Princeton, N.J., 1956 and S. MacLane, *Homology,* Springer-Verlag OHG, Berlin, 1963.

4 *For R modules A and B there is an isomorphism of $A \otimes B$ with $B \otimes A$ taking $a \otimes b$ to $b \otimes a$.*

5 *If A and B are R modules and B and C are R′ modules, there is an isomorphism of $(A \underset{R}{\otimes} B) \underset{R'}{\otimes} C$ with $A \underset{R}{\otimes} (B \underset{R'}{\otimes} C)$ (both being regarded as R and R′ modules) taking $(a \otimes b) \otimes c$ to $a \otimes (b \otimes c)$.*

If A and B are R modules, their *module of homomorphisms* Hom (A,B) [also written $\mathrm{Hom}_R\,(A,B)$] is an R module whose elements are R homomorphisms $A \to B$. In case A or B is also an R' module, then so is $\mathrm{Hom}_R\,(A,B)$.

6 *If A and B are R modules and B and C are R′ modules, there is an isomorphism of* $\mathrm{Hom}_{R'}\,(A \underset{R}{\otimes} B,\ C)$ *with* $\mathrm{Hom}_R\,(A,\ \mathrm{Hom}_{R'}\,(B,C))$ *(both being regarded as R and R′ modules) taking an R′ homomorphism $\varphi\colon A \underset{R}{\otimes} B \to C$ to the R homomorphism $\varphi'\colon A \to \mathrm{Hom}_{R'}\,(B,C)$ such that $\varphi'(a)(b) = \varphi(a \otimes b)$.*

A subset S of an R module A is said to be a *basis for A* (and A is said to be *freely generated by* S) if any function $f\colon S \to B$, where B is an R module, admits a unique extension to a homomorphism $\varphi\colon A \to B$. If a module has a basis, it is said to be a *free module*. For any set S the *free module generated by* S, denoted by $F_R(S)$, is the module of all finitely nonzero functions from S to R (with pointwise addition and scalar multiplication) and with $s \in S$ identified with its characteristic function. $F_R(S)$ contains S as a basis, and any module containing S as a basis is canonically isomorphic to $F_R(S)$.

7 *Any R module is isomorphic to a quotient of a free R module.*

8 *If A′ is a submodule of A, with A/A′ free, then A is isomorphic to the direct sum $A' \oplus (A/A')$.*

We now assume that R is a principal ideal domain (that is, it is an integral domain in which every ideal is principal). If A is an R module, its *torsion submodule* Tor A is defined by

$$\mathrm{Tor}\,A = \{a \in A \mid ra = 0 \text{ for some nonzero } r \in R\}$$

A is said to be *torsion free* or *without torsion* if Tor $A = 0$.

9 *Over a principal ideal domain, a submodule of a free module is free.*

10 *Over a principal ideal domain, a finitely generated module is free if and only if it is torsion free.*

11 *Over a principal ideal domain, A/*Tor *A is torsion free.*

If A is a finitely generated module over a principal ideal domain R, its *rank* $\rho(A)$ is defined to be the number of elements in a basis of the quotient module $A/\mathrm{Tor}\,A$.

12 *If A′ is a submodule of a finitely generated module A (over a principal ideal domain), then*

$$\rho(A) = \rho(A') + \rho(A/A')$$

Let $\varphi\colon A \to A$ be an endomorphism of a finitely generated module (over a principal ideal domain R). The *trace* of φ, $\operatorname{Tr} \varphi$, is the element of R which is the trace of the endomorphism φ' induced by φ on the free module $A/\operatorname{Tor} A$ [that is, if $A/\operatorname{Tor} A$ has a basis $a_1, \ldots, a_n$, then $\varphi'(a_i) = \sum r_{ij}a_j$ and $\operatorname{Tr} \varphi = \sum r_{ii}$].

13 *Let φ be an endomorphism of a finitely generated module A and let A' be a submodule of A such that $\varphi(A') \subset A'$. Then $\varphi \mid A'$ is an endomorphism of A' and there is induced an endomorphism φ'' of A/A'. Their traces satisfy the relation*

$$\operatorname{Tr} \varphi = \operatorname{Tr} (\varphi \mid A') + \operatorname{Tr} \varphi''$$

A module with a single generator is said to be *cyclic*. Over a principal ideal domain R such a module A is characterized, up to isomorphism, by the element $r_A \in R$ which generates the ideal of elements annihilating every element of A (r_A is unique up to multiplication by invertible elements of R).

14 STRUCTURE THEOREM FOR FINITELY GENERATED MODULES *Over a principal ideal domain every finitely generated module is the direct sum of a free module and cyclic modules $A_1, \ldots, A_q$ whose corresponding elements $r_1, \ldots, r_q \in R$ have the property that r_i divides r_{i+1} for $1 \le i \le q - 1$. The elements $r_1, \ldots, r_q$ are unique up to multiplication by invertible elements of R and, together with the rank of the module, characterize the module up to isomorphism.*

5 EUCLIDEAN SPACES

We use the following fixed notations:

$\varnothing$ = empty set
$\mathbf{Z}$ = ring of integers
$\mathbf{Z}_m$ = ring of integers modulo m
$\mathbf{R}$ = field of real numbers
$\mathbf{C}$ = field of complex numbers
$\mathbf{Q}$ = division ring of quaternions
$\mathbf{R}^n$ = *euclidean n-space*, with $\|x\| = \sqrt{\sum x_i^2}$ and $\langle x,y\rangle = \sum x_i y_i$
0 = *origin of* $\mathbf{R}^n$
I = *closed unit interval*
$\dot I$ = $\{0,1\} \subset I$
I^n = *n-cube* = $\{x \in \mathbf{R}^n \mid 0 \le x_i \le 1 \text{ for } 1 \le i \le n\}$
$\dot I^n$ = $\{x \in I^n \mid \text{for some } i,\ x_i = 0 \text{ or } x_i = 1\}$
E^n = *n-ball* = $\{x \in \mathbf{R}^n \mid \|x\| \le 1\}$
S^{n-1} = *$(n-1)$-sphere* = $\{x \in \mathbf{R}^n \mid \|x\| = 1\}$
P^n = *projective n-space* = quotient space of S^n with x and $-x$ identified for all $x \in S^n$

If x and y are points of a real vector space, the *closed line segment* joining them, denoted by $[x,y]$, is the set of points of the form $tx + (1 - t)y$ for $0 \leq t \leq 1$ (thus $I = [0,1]$). If $x \neq y$, the *line* determined by them is the set $\{tx + (1 - t)y \mid t \in \mathbf{R}\}$. A subset C of a real vector space is said to be an *affine variety* if whenever $x, y \in C$, with $x \neq y$, then the line determined by x and y is also in C. A subset C is said to be *convex* if $x, y \in C$ imply $[x,y] \subset C$. A *convex body*[1] in $\mathbf{R}^n$ is a convex subset of $\mathbf{R}^n$ containing a nonempty open subset of $\mathbf{R}^n$ (thus I^n and E^n are convex bodies in $\mathbf{R}^n$).

1 *If C is a convex body in $\mathbf{R}^n$ and C' is a convex body in $\mathbf{R}^m$, then $C \times C'$ is a convex body in $\mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^{n+m}$.*

2 *Any two compact convex bodies in $\mathbf{R}^n$ are homeomorphic.*

A subset S of a real vector space is said to be *affinely independent* if, given a finite number of distinct elements $x_0, x_1, \ldots, x_m \in S$ and $t_0, t_1, \ldots, t_m \in \mathbf{R}$ such that $\sum t_i = 0$ and $\sum t_i x_i = 0$, then $t_i = 0$ for $0 \leq i \leq m$ (this is equivalent to the condition that

$$x_1 - x_0, x_2 - x_0, \ldots, x_m - x_0$$

be linearly independent).

3 *There exist affinely independent subsets of $\mathbf{R}^n$ containing $n + 1$ points, but no subset of $\mathbf{R}^n$ containing more than $n + 1$ points is affinely independent.*

4 *Given points $x_0, x_1, \ldots, x_m \in \mathbf{R}^n$, the convex set generated by them is the set of all points of the form $\sum t_i x_i$, with $0 \leq t_i \leq 1$ and $\sum t_i = 1$. The set $\{x_0, x_1, \ldots, x_m\}$ is affinely independent if and only if every point x in the convex set generated by this set has a unique representation in the form $x = \sum t_i x_i$, with $0 \leq t_i \leq 1$ for $0 \leq i \leq m$ and $\sum t_i = 1$.*

SOME BOOKS ON ALGEBRAIC TOPOLOGY

Alexandroff, P. and H. Hopf: *Topologie*, Springer-Verlag, 1935.

Bott, R. and L.W. Tu: *Differential Forms in Algebraic Topology*, Springer-Verlag, 1982

Bourgin, D.G.: *Modern Algebraic Topology*, Macmillan, 1963.

Bredon, G.E.: *Sheaf Theory*, McGraw-Hill, 1967.

Cairns, S.S.: *Introductory Topology*, Ronald Press, 1962.

Dold, A.: *Lectures on Algebraic Topology*, Springer-Verlag, 1980.

Eilenberg, S. and N.E. Steenrod: *Foundations of Algebraic Topology*, Princeton University Press, 1952.

Godement, R.: *Topologie algébrique et théorie des faisceaux*, Hermann and Cie, 1958.

[1] For general properties of convex sets see F. A. Valentine, *Convex Sets*, McGraw-Hill Book Company, New York, 1964.

Gray, B.: *Homotopy Theory, An Introduction to Algebraic Topology*, Academic Press, 1975.

Greenberg, M.J. and J.R. Harper: *Algebraic Topology, A First Course* Benjamin/Cummings, 1981.

Hilton, P.J. and S. Wylie: *Homology Theory*, Cambridge University Press, 1960.

Hocking, J.G. and G.S. Young: *Topology*, Addison-Wesley, 1961.

Hu, S.T.: *Homotopy Theory*, Academic Press, 1959.

Lefschetz, S.: *Algebraic Topology*, American Math Society, 1942.

Lefschetz, S.: *Introduction to Topology*, Princeton Univ. Press, 1949.

Massey, W.S.: *Algebraic Topology, An Introduction*, Harcourt, Brace and World, 1967.

Massey, W.S.: *Homology and Cohomology Theory, An Approach Based on Alexander-Spanier Cochains*, Dekker, 1978.

Massey, W.S.: *Singular Homology Theory*, Springer-Verlag, 1980.

Maunder, C.R.F.: *Algebraic Topology*, Cambridge Univ. Press, 1980.

Munkres, J.R.: *Elements of Algebraic Topology*, Addison-Wesley, 1984.

Pontryagin, L.S.: *Foundations of Combinational Topology*, Graylock Press, 1952.

Schubert H.: *Topologie*, Teubner Verlagsgesellschaft, 1964.

Seifert, H. and W. Threlfall: *Lehrbuch der Topologie*, Teubner Verlagsgesellschaft, 1934.

Steenrod, N.E.: *The Topology of Fiber Bundles*, Princeton University Press, 1951.

Switzer, R.M.: *Algebraic Topology—Homotopy and Homology*, Springer-Verlag, 1975.

Vick, J.W.: *Homology Theory, An Introduction to Algebraic Topology*, Academic Press, 1973.

Wallace, A.H.: *An Introduction to Algebraic Topology*, Pergamon Press, 1957.

Wallace, A.H.: *Algebraic Topology, Homology and Cohomology*, Benjamin, 1970.

Wilder, R.L.: *Topology of Manifolds*, American Math Society, 1949.

CHAPTER ONE

HOMOTOPY AND THE FUNDAMENTAL GROUP

TOPOLOGY IS THE STUDY OF TOPOLOGICAL SPACES AND CONTINUOUS FUNCTIONS between them. A standard problem is the classification of such spaces and functions up to homeomorphism. A weaker equivalence relation, based on continuous deformation, leads to another classification problem. This latter classification problem is of fundamental importance in algebraic topology, since it is the one where the tools available seem to be most successful.

As a working definition for our purposes, algebraic topology may be regarded as the study of topological spaces and continuous functions by means of algebraic objects such as groups, rings, homomorphisms. The link from topology to algebra is by means of mappings, called functors. For this reason, Secs. 1.1 and 1.2 are devoted to the basic concepts of category and functor.

In Secs. 1.3 and 1.4 the concept of continuous deformation, known technically as homotopy, is introduced. We then define the homotopy category and certain functors on this category, all of which are important for the subject. Sections 1.5 and 1.6 are devoted to a study of conditions under which these functors on the homotopy category take values in the category of groups. As examples, the homotopy group functors are briefly mentioned.

The first functor considered in detail is the fundamental group functor, introduced and discussed in Secs. 1.7 and 1.8. This is an intuitively appealing example of the kind of functor considered in algebraic topology. Some applications of this functor are presented in the exercises at the end of the chapter. In Chapter Two this functor is used in a systematic study and classification of covering spaces.

1 CATEGORIES

An algebraic representation of topology is a mapping from topology to algebra. Such a representation converts a topological problem into an algebraic one to the end that, with sufficiently many representations, the topological problem will be solvable if (and only if) all the corresponding algebraic problems are solvable.

The definition of a representation, formally called a functor, is given in the next section. This section is devoted to the concept of category, because functors are functions, with certain naturality properties, from one or several categories to another.

A category may be thought of intuitively as consisting of sets, possibly with additional structure, and functions, possibly preserving additional structure. More precisely, a *category* $\mathcal{C}$ consists of

(*a*) A class of *objects*
(*b*) For every ordered pair of objects X and Y, a set hom (X,Y) of *morphisms* with *domain* X and *range* Y; if $f \in$ hom (X,Y), we write $f\colon X \to Y$ or $X \xrightarrow{f} Y$
(*c*) For every ordered triple of objects X, Y, and Z, a function associating to a pair of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ their *composite*

$$gf = g \circ f\colon X \to Z$$

These satisfy the following two axioms:

Associativity. If $f\colon X \to Y$, $g\colon Y \to Z$, and $h\colon Z \to W$, then

$$h(gf) = (hg)f\colon X \to W$$

Identity. For every object Y there is a morphism $1_Y\colon Y \to Y$ such that if $f\colon X \to Y$, then $1_Y f = f$, and if $h\colon Y \to Z$, then $h1_Y = h$.

If the class of objects is a set, the category is said to be *small.* For most of our purposes we could restrict our attention to small categories, but it would be inconvenient to have to specify a set of objects before obtaining a category. For example, we should like to consider categories whose objects are sets or groups, and we prefer to consider the class of all sets or groups, rather than some suitable set of sets or groups in each instance.

From the two axioms it follows that 1_Y is unique (see lemma 1 below), and it is called the *identity morphism* of Y. Given morphisms $f\colon X \to Y$ and $g\colon Y \to X$ such that $gf = 1_X$, g is called a *left inverse* of f and f is called a *right inverse* of g. A *two-sided inverse* (or simply an *inverse*) of f is a morphism which is both a left inverse of f and a right inverse of f. A morphism $f\colon X \to Y$ is called an *equivalence*, denoted by $f\colon X \approx Y$, if there is a morphism $g\colon Y \to X$ which is a two-sided inverse of f. If $g'\colon Y \to X$ is a left inverse of f and $g''\colon Y \to X$ is a right inverse of f, then

$$g' = g'1_Y = g'(fg'') = (g'f)g'' = 1_X g'' = g''$$

showing that $g' = g''$. Therefore we have the following lemma.

1 LEMMA *If $f\colon X \to Y$ has a left inverse and a right inverse, they are equal, and f is an equivalence.* ∎

In particular, it follows that an equivalence $f\colon X \approx Y$ has a unique inverse, denoted by $f^{-1}\colon Y \to X$, and f^{-1} is an equivalence. If there is an equivalence $f\colon X \approx Y$, X and Y are said to be *equivalent*, denoted by $X \approx Y$. Because the composite of equivalences is easily seen to be an equivalence, the relation $X \approx Y$ is an equivalence relation in any set of objects of $\mathcal{C}$.

We list some examples of categories.

2 The category of sets and functions [that is, the class of objects is the class of all sets, and for sets X and Y, hom (X,Y) equals the set of functions from X to Y]

3 The category of topological spaces and continuous maps

4 The category of groups and homomorphisms

5 The category of R modules and homomorphisms

6 The category of normed rings (over $\mathbf{R}$) and continuous homomorphisms

7 The category of sets and injections (or surjections or bijections)

8 The category of *pointed sets* (a pointed set is a nonempty set with a distinguished element) and functions preserving distinguished elements

9 The category of *pointed topological spaces* (a pointed topological space is a nonempty topological space with a *base point*) and continuous maps preserving base points

10 The category of finite sets and functions

11 Given a partial order $\le$ in X, there is a category whose objects are the elements of X and such that hom (x,x') is either the singleton consisting of the ordered pair (x,x') or empty, according to whether $x \le x'$ or $x \not\le x'$

12 The category of groups and conjugacy classes of homomorphisms (that is, a morphism $G \to G'$ is an equivalence class of homomorphisms from G to G', two homomorphisms being equivalent if they differ by an inner automorphism of G')

A *subcategory* $\mathcal{C}' \subset \mathcal{C}$ is a category such that

(*a*) The objects of $\mathcal{C}'$ are also objects of $\mathcal{C}$
(*b*) For objects X' and Y' of $\mathcal{C}'$, $\hom_{\mathcal{C}'}(X',Y') \subset \hom_{\mathcal{C}}(X',Y')$
(*c*) If $f'\colon X' \to Y'$ and $g'\colon Y' \to Z'$ are morphisms of $\mathcal{C}'$, their composite in $\mathcal{C}'$ equals their composite in $\mathcal{C}$

$\mathcal{C}'$ is called a *full subcategory* of $\mathcal{C}$ if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ and for objects X' and Y' in $\mathcal{C}'$, $\hom_{\mathcal{C}'}(X',Y') = \hom_{\mathcal{C}}(X',Y')$. The category in example 7 above is a subcategory of the one in example 2, and the category in example 10 is a full subcategory of the one in example 2. The categories in examples 3, 4, 5, 6, and 8 are not subcategories of the category of sets, because each object of one of these categories consists of a set, together with an additional structure on it (hence, different objects in these categories may have the same underlying sets). In examples 11 and 12, the morphisms in the respective categories are not functions, and so neither of these categories is a subcategory of the category of sets.

A diagram of morphisms such as the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

is said to be *commutative* if any two composites of morphisms in the diagram beginning at the same place and ending at the same place are equal. This square is commutative if and only if $hf = f'g$.

Following are descriptions of some categories which are associated to a given category. Given a category $\mathcal{C}$, there is an associated category called the *category of morphisms of* $\mathcal{C}$. Its objects are morphisms $X \xrightarrow{f} Y$, and its morphisms with domain $X \xrightarrow{f} Y$ and range $X' \xrightarrow{f'} Y'$ are pairs of morphisms $g\colon X \to X'$ and $h\colon Y \to Y'$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

is commutative. In a similar way, diagrams of morphisms in $\mathcal{C}$ more general than $X \xrightarrow{f} Y$ are the objects of a suitable category associated to $\mathcal{C}$.

Let $\mathcal{C}$ be a category whose objects are sets with additional structures (such as distinguished elements or topologies) and whose morphisms are functions preserving the additional structures. For example, $\mathcal{C}$ might be any of the categories in examples 2 through 10. There is a category associated to $\mathcal{C}$, called the *category of pairs of* $\mathcal{C}$, whose objects are injective morphisms $i\colon A \to X$ (because each morphism in such a category is a function, it is meaningful to consider those which are injective) and whose morphisms are commutative squares

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{j} & Y \end{array}$$

Thus the category of pairs of $\mathcal{C}$ is a full subcategory of the category of morphisms of $\mathcal{C}$. The notation (X,A) will denote the pair consisting of X and i: $A \subset X$, and the notation f: $(X,A) \to (Y,B)$ will mean that f: $X \to Y$ is a morphism of $\mathcal{C}$ such that $f(i(A)) \subset j(B)$. The category of pairs of $\mathcal{C}$, therefore, has as objects the pairs (X,A) and has as morphisms the morphisms f: $(X,A) \to (Y,B)$.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are categories, their *product* $\mathcal{C}_1 \times \mathcal{C}_2$ is the category whose objects are ordered pairs (Y_1, Y_2) of objects Y_1 in $\mathcal{C}_1$ and Y_2 in $\mathcal{C}_2$ and whose morphisms $(X_1,X_2) \to (Y_1, Y_2)$ are ordered pairs of morphisms (f_1,f_2), where f_1: $X_1 \to Y_1$ in $\mathcal{C}_1$ and f_2: $X_2 \to Y_2$ in $\mathcal{C}_2$. Similarly, there is a product of an arbitrary indexed family of categories.

Given a category $\mathcal{C}$, there is an *opposite* category $\mathcal{C}^*$ whose objects Y^* are in one-to-one correspondence with the objects Y of $\mathcal{C}$ and whose morphisms f^*: $Y^* \to X^*$ are in one-to-one correspondence with the morphisms f: $X \to Y$ [with $f^* g^*$ defined to equal $(gf)^*$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$]. We identify $(\mathcal{C}^*)^*$ with $\mathcal{C}$, so that $(X^*)^* = X$ and $(f^*)^* = f$.

We next show how to interpret sums and products, as well as direct and inverse limits in arbitrary categories. An object X in a category $\mathcal{C}$ is said to be an *initial object* if for each object Y in $\mathcal{C}$ the set hom (X,Y) contains exactly one element. Dually, an object Z of $\mathcal{C}$ is said to be a *terminal object* if for each Y of $\mathcal{C}$ the set hom (Y,Z) contains exactly one element. Note that any two initial objects of $\mathcal{C}$ are equivalent and any two terminal objects of $\mathcal{C}$ are equivalent. In examples 2 and 3 the empty set is an initial object and any one-point set is a terminal object. In example 4 the trivial group is both an initial and a terminal object. In example 7 the category of sets and bijections has neither an initial object nor a terminal object.

Let $\{Y_j\}_{j \in J}$ be an indexed collection of objects of a category $\mathcal{C}$. Let $\mathcal{S}\{Y_j\}$ be the category whose objects are indexed collections of morphisms $\{f_j\}_{j \in J}$ of $\mathcal{C}$ having the same range and whose morphisms with domain $\{f_j$: $Y_j \to Z\}$ and range $\{f_j'$: $Y_j \to Z'\}$ are morphisms g: $Z \to Z'$ of $\mathcal{C}$ such that $gf_j = f_j'$ for every $j \in J$. An initial object of $\mathcal{S}\{Y_j\}$ is called a *sum* of the collection $\{Y_j\}$. A given collection may or may not have a sum in $\mathcal{C}$. The set sum is a sum in the category of sets, the topological sum is a sum in the category of topological spaces, the free product is a sum in the category of groups, and the direct sum is a sum in the category of R modules. In the category of finite sets, in general only finite collections have a sum. Similarly, in the category of finitely generated R modules, in general only finite collections have a sum.

Dually, given an indexed collection of objects $\{Y_j\}_{j \in J}$ in $\mathcal{C}$, let $\mathcal{P}\{Y_j\}$ be the category whose objects are indexed collections of morphisms $\{g_j\}_{j \in J}$ of $\mathcal{C}$ having the same domain and whose morphisms with domain $\{g_j$: $X \to Y_j\}$ and range $\{g_j'$: $X' \to Y_j\}$ are morphisms f: $X \to X'$ of $\mathcal{C}$ such that $g_j'f = g_j$ for every $j \in J$. A terminal object of $\mathcal{P}\{Y_j\}$ is called a *product* of the collection $\{Y_j\}$. The cartesian product of sets is a product in the category of sets, the topological product is a product in the category of topological spaces, and the direct product is a product in the category of groups, or R modules. In the category of finite sets (or finitely generated R modules), in general only finite collections have a product.

A *direct system* $\{Y^\alpha, f_\alpha{}^\beta\}$ in a category $\mathcal{C}$ consists of a collection of objects $\{Y^\alpha\}$ indexed by a directed set $\Lambda = \{\alpha\}$ and a collection of morphisms $\{f_\alpha{}^\beta: Y^\alpha \to Y^\beta\}$ in $\mathcal{C}$ for $\alpha \le \beta$ in Λ such that

(a) $f_\alpha{}^\alpha = 1_{Y^\alpha}$ for $\alpha \in \Lambda$
(b) $f_\alpha{}^\gamma = f_\beta{}^\gamma f_\alpha{}^\beta: Y^\alpha \to Y^\gamma$ for $\alpha \le \beta \le \gamma$ in Λ

There is then a category dir $\{Y^\alpha, f_\alpha{}^\beta\}$ whose objects are indexed collections of morphisms $\{g_\alpha: Y^\alpha \to Z\}_{\alpha \in \Lambda}$ such that $g_\alpha = g_\beta f_\alpha{}^\beta$ if $\alpha \le \beta$ in Λ and whose morphisms with domain $\{g_\alpha: Y^\alpha \to Z\}$ and range $\{g'_\alpha: Y^\alpha \to Z'\}$ are morphisms $h: Z \to Z'$ such that $hg_\alpha = g'_\alpha$ for $\alpha \in \Lambda$. An initial object of dir $\{Y^\alpha, f_\alpha{}^\beta\}$ is called a *direct limit* of the direct system $\{Y^\alpha, f_\alpha{}^\beta\}$. The direct limits of sets, topological spaces, groups, and R modules are examples of direct limits in their respective categories.

Dually, an *inverse system* $\{Y_\alpha, f_\alpha{}^\beta\}$ in $\mathcal{C}$ consists of a collection of objects $\{Y_\alpha\}$ indexed by a directed set $\Lambda = \{\alpha\}$ and a collection of morphisms $\{f_\alpha{}^\beta: Y_\beta \to Y_\alpha\}$ in $\mathcal{C}$ for $\alpha \le \beta$ in Λ such that

(a) $f_\alpha{}^\alpha = 1_{Y_\alpha}$ for $\alpha \in \Lambda$
(b) $f_\alpha{}^\gamma = f_\alpha{}^\beta f_\beta{}^\gamma: Y_\gamma \to Y_\alpha$ for $\alpha \le \beta \le \gamma$ in Λ

There is then a category inv $\{Y_\alpha, f_\alpha{}^\beta\}$ whose objects are indexed collections of morphisms $\{g_\alpha: X \to Y_\alpha\}_{\alpha \in \Lambda}$ such that $g_\alpha = f_\alpha{}^\beta g_\beta$ if $\alpha \le \beta$ in Λ and whose morphisms with domain $\{g_\alpha: X \to Y_\alpha\}$ and range $\{g'_\alpha: X' \to Y_\alpha\}$ are morphisms $h: X \to X'$ of $\mathcal{C}$ such that $g'_\alpha h = g_\alpha$ for $\alpha \in \Lambda$. A terminal object of inv $\{Y_\alpha, f_\alpha{}^\beta\}$ is called an *inverse limit* of the inverse system $\{Y_\alpha, f_\alpha{}^\beta\}$. The inverse limits of sets, topological spaces, groups, and R modules are examples of inverse limits in their respective categories.

By similar considerations it is possible to define a direct or inverse limit for an arbitrary indexed collection of objects in a category $\mathcal{C}$ and an indexed collection of morphisms in $\mathcal{C}$ between these objects. We omit the details.

2 FUNCTORS

Our main interest in categories is in the maps from one category to another. Those maps which have the natural properties of preserving identities and composites are called functors. This section is devoted to the definition of functors of one or more variables, some examples and applications, and the definition of natural transformations between functors.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* (or *contravariant functor*) T from $\mathcal{C}$ to $\mathcal{D}$ consists of an object function which assigns to every object X of $\mathcal{C}$ an object $T(X)$ of $\mathcal{D}$ and a morphism function which assigns to every morphism $f: X \to Y$ of $\mathcal{C}$ a morphism $T(f): T(X) \to T(Y)$ [or $T(f): T(Y) \to T(X)$] of $\mathcal{D}$ such that

(a) $T(1_X) = 1_{T(X)}$
(b) $T(gf) = T(g)T(f)$ [or $T(gf) = T(f)T(g)$]

We list some examples of functors.

1 There is a covariant functor from the category of topological spaces and continuous maps to the category of sets and functions which assigns to every topological space its underlying set. This functor is called a *forgetful functor* because it "forgets" some of the structure of a topological space.

2 There is a covariant functor from the category of sets and functions to the category of R modules and homomorphisms which assigns to every set the free R module generated by it.

3 Given a fixed R module M_0, there is a covariant functor (or contravariant functor) from the category of R modules and homomorphisms to itself which assigns to an R module M the R module $\operatorname{Hom}_R(M_0,M)$ [or $\operatorname{Hom}_R(M,M_0)$].

4 For any category $\mathcal{C}$ and object Y of $\mathcal{C}$ there is a covariant functor π_Y (or contravariant functor π^Y) from $\mathcal{C}$ to the category of sets and functions which assigns to an object Z (or X) of $\mathcal{C}$ the set $\pi_Y(Z) = \hom(Y,Z)$ [or $\pi^Y(X) = \hom(X,Y)$] and to a morphism $h: Z \to Z'$ [or $f: X \to X'$] the function

$$h_\#: \hom(Y,Z) \to \hom(Y,Z') \qquad [\text{or } f^\#: \hom(X',Y) \to \hom(X,Y)]$$

defined by $h_\#(g) = h \circ g$ for $g: Y \to Z$ [or $f^\#(g') = g' \circ f$ for $g': X' \to Y$]

5 There is a contravariant functor C from the category of compact Hausdorff spaces and continuous maps to the category of normed rings over $\mathbf{R}$ and continuous homomorphisms which assigns to X its normed ring of continuous real-valued functions.

6 There is a covariant functor H_0 from the category of topological spaces and continuous maps to the category of abelian groups and homomorphisms such that $H_0(X)$ is the free abelian group generated by the set of components of X, and if $f: X \to Y$, then $H_0(f): H_0(X) \to H_0(Y)$ is the homomorphism such that if C is a component of X and C' is the component of Y containing $f(C)$, then $H_0(f)C = C'$.

7 A direct system (or inverse system) in a category $\mathcal{C}$ is a covariant functor (or contravariant functor) from the category of a directed set (defined as in example 1.1.11) to $\mathcal{C}$.

8 For any category $\mathcal{C}$ there is a contravariant functor to its opposite category $\mathcal{C}^*$ which assigns to an object X of $\mathcal{C}$ the object X^* of $\mathcal{C}^*$ and to a morphism $f: X \to Y$ of $\mathcal{C}$ the morphism $f^*: Y^* \to X^*$.

Note that any contravariant functor on $\mathcal{C}$ corresponds to a covariant functor on $\mathcal{C}^*$, and vice versa. Therefore any functor can be regarded as covariant on a suitable category. Despite this, we shall find it convenient to consider contravariant as well as covariant functors on $\mathcal{C}$, rather than consider only covariant functors on two categories.

Any functor from the category of topological spaces and continuous maps to an algebraic category (such as the category of abelian groups and homomorphisms) is a representation of the topological category by an algebraic one. Algebraic topology is the study of such functors; we show that simple remarks about functors can be used to obtain necessary conditions for the solvability of topological problems.

9 **THEOREM** *Let T be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$. Then T maps equivalences in $\mathcal{C}$ to equivalences in $\mathcal{D}$.*

PROOF Assume that T is a covariant functor (the argument is similar if T is contravariant). Let $f\colon X \to Y$ be an equivalence in $\mathcal{C}$. Then $f^{-1}f = 1_X$. Therefore

$$1_{T(X)} = T(1_X) = T(f^{-1})T(f)$$

Similarly, $T(f)T(f^{-1}) = 1_{T(Y)}$. Therefore $T(f^{-1})$ is a two-sided inverse of $T(f)$, and $T(f)$ is an equivalence in $\mathcal{D}$. ∎

In particular, if T is an algebraic functor on the category of topological spaces and continuous maps, a necessary condition that X be homeomorphic to Y is that $T(X)$ be equivalent to $T(Y)$. Thus the functor H_0 of example 6 shows that the real line $\mathbf{R}$ and the real plane $\mathbf{R}^2$ are not homeomorphic [if they were homeomorphic, then $\mathbf{R} - 0$ would be homeomorphic to $\mathbf{R}^2 - p$ for some $p \in \mathbf{R}^2$, but $H_0(\mathbf{R} - 0)$ is a free abelian group on two generators, while $H_0(\mathbf{R}^2 - p)$ is a free abelian group on one generator]. This is a trivial example. However, the homology functors H_q defined in Chapter 4 generalize H_0 and can be used in much the same way to prove that $\mathbf{R}^n$ and $\mathbf{R}^m$ are not homeomorphic if $n \neq m$.

In applications of algebraic functors to topological problems the algebra will frequently play an essential role. For example, let $T_0(X)$ be the functor obtained by composing the functor H_0 with the forgetful functor, which assigns to every abelian group its underlying set. The functor T_0 contains less information than the functor H_0 and does not give as strong a necessary condition for homeomorphism [for example, $T_0(\mathbf{R} - 0)$ and $T_0(\mathbf{R}^2 - p)$ are both countably infinite sets and are equivalent in the category of sets and functions]. For this reason it is important to provide functors with as much algebraic structure as possible. Later we shall consider functors which depend on a chosen topological space. These functors take values in the category of sets and functions, but some of them, depending on properties of the particular spaces which define them, are functors to the category of groups and homomorphisms. The added algebraic structure in such cases will prove useful.

To show how functors can be applied to another problem, let A be a subspace of a topological space X and let $f\colon A \to Y$ be continuous. The *extension problem* is to determine whether f has a continuous extension to X—that is, whether the dotted arrow in the triangle

$$\begin{tikzcd} A \arrow[r, phantom, "\subset"] \arrow[dr, "f"'] & X \arrow[d, dotted] \\ & Y \end{tikzcd}$$

corresponds to a continuous map making the diagram commutative.

10 THEOREM *Let T be a covariant functor (or contravariant functor) from the category of topological spaces and continuous maps to a category $\mathcal{C}$. A necessary condition that a map $f: A \to Y$ be extendable to X (where $i: A \subset X$) is that there exist a morphism $\varphi: T(X) \to T(Y)$ [or $\varphi: T(Y) \to T(X)$] such that $\varphi \circ T(i) = T(f)$ [or $T(f) = T(i) \circ \varphi$].*

PROOF Assume that $f': X \to Y$ is an extension of f. Then $f'i = f$. Therefore $T(f') \circ T(i) = T(f)$ [or $T(f) = T(i) \circ T(f')$], and $T(f')$ can be taken as the morphism φ. ∎

The above result can be applied to prove that the identity map of $\dot{I}$ cannot be extended to a continuous map $I \to \dot{I}$. We use the functor H_0 and obtain the necessary condition that there must exist a homomorphism $\varphi: H_0(I) \to H_0(\dot{I})$ such that $\varphi \circ H_0(i) = H_0(1_{\dot{I}})$ (where $i: \dot{I} \subset I$). Because $H_0(\dot{I})$ is a free abelian group on two generators and $H_0(I)$ is a free abelian group on one generator, there is no such homomorphism φ. Again, this is a trivial example, but it illustrates the method, and the general homology functors H_q defined later can be used in the same way to show that there is no continuous map $E^{n+1} \to S^n$ that is the identity map on S^n.

Thus we see that a functor yields necessary conditions for the solvability of topological problems. There are situations in which these necessary conditions are also sufficient. For example, the functor C of example 5 gives a necessary and sufficient condition for homeomorphism—that is, two compact Hausdorff spaces X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are isomorphic.[1] This is not a particularly useful result, however, because it seems to be no easier to determine whether or not two normed rings are isomorphic than it is to determine whether or not two compact Hausdorff spaces are homeomorphic. We seek functors to categories that are somewhat simpler than the category of topological spaces, so that the algebraic problems that arise in these categories can be effectively solved. One big problem of algebraic topology is to find, and compute, sufficiently many such functors that the solvability of a particular topological problem is equivalent to the solvability of the corresponding (and simpler) algebraic problems.

We shall also have occasion to compare functors with each other. This is done by means of a suitable definition of a map between functors. Let T_1 and T_2 be functors of the same variance (either both covariant or both contravariant) from a category $\mathcal{C}$ to a category $\mathcal{D}$. A *natural transformation* φ from T_1 to T_2 is a function from the objects of $\mathcal{C}$ to morphisms of $\mathcal{D}$ such that for every morphism $f: X \to Y$ of $\mathcal{C}$ the appropriate one of the following diagrams is commutative:

$$\begin{array}{ccc} T_1(X) & \xrightarrow{T_1(f)} & T_1(Y) \\ {\scriptstyle\varphi(X)}\downarrow & & \downarrow{\scriptstyle\varphi(Y)} \\ T_2(X) & \xrightarrow{T_2(f)} & T_2(Y) \end{array} \qquad \begin{array}{ccc} T_1(X) & \xleftarrow{T_1(f)} & T_1(Y) \\ {\scriptstyle\varphi(X)}\downarrow & & \downarrow{\scriptstyle\varphi(Y)} \\ T_2(X) & \xleftarrow{T_2(f)} & T_2(Y) \end{array}$$

T_1, T_2 *covariant* $\qquad$ T_1, T_2 *contravariant*

[1] See Theorem D on page 330 of G. F. Simmons, *Introduction to Topology and Modern Analysis*, McGraw-Hill Book Company, New York, 1963.

If φ is a natural transformation from T_1 to T_2 such that $\varphi(X)$ is an equivalence in $\mathcal{D}$ for each object X in $\mathcal{C}$, then φ is called a *natural equivalence.*

As an example of a natural transformation, let Y_1 and Y_2 be objects of a category $\mathcal{C}$ and let $g\colon Y_1 \to Y_2$ be a morphism in $\mathcal{C}$. There is a natural transformation $g^\#$ from the covariant functor π_{Y_2} to the covariant functor π_{Y_1} and a natural transformation $g_\#$ from the contravariant functor π^{Y_1} to the contravariant functor π^{Y_2}. If g is an equivalence in $\mathcal{C}$, both these natural transformations are natural equivalences.

It is also of interest to consider functors of several variables. Thus, if $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{D}$ are categories, a covariant functor from $\mathcal{C}_1 \times \mathcal{C}_2$ to $\mathcal{D}$ is called a *functor of two arguments covariant* in each. A covariant functor from $\mathcal{C}_1 \times \mathcal{C}_2^*$ to $\mathcal{D}$, regarded as a function from ordered pairs (X_1,X_2), where X_1 is an object of $\mathcal{C}_1$ and X_2 is an object of $\mathcal{C}_2$, is called a *functor of two arguments covariant* in the first and *contravariant* in the second. In a similar fashion, functors of more arguments with mixed variance are defined.

If $\mathcal{C}$ is any category, there is a functor of two arguments in $\mathcal{C}$ to the category of sets and functions which is contravariant in the first argument and covariant in the second. This functor assigns to an ordered pair of objects X and Y of $\mathcal{C}$ the set hom (X,Y) and to an ordered pair of morphisms $f\colon X' \to X$ and $g\colon Y \to Y'$ in $\mathcal{C}$ the function $f^\# g_\# = g_\# f^\#\colon \hom(X,Y) \to \hom(X',Y')$.

3 HOMOTOPY

The problem of classifying topological spaces and continuous maps up to topological equivalence does not seem to be amenable to attack directly by computable algebraic functors, as described in Sec. 1.2. Many of the computable functors, because they are computable, are invariant under continuous deformation. Therefore they cannot distinguish between spaces (or maps) that can be continuously deformed from one to the other; the most that can be hoped for from such functors is that they characterize the space (or map) up to continuous deformation.

The intuitive concept of a continuous deformation will be made precise in this section in the concept of homotopy. This leads to the homotopy category which is fundamental for algebraic topology. Its objects are topological spaces and its morphisms are equivalence classes of continuous maps (two maps being equivalent if one can be continuously deformed into the other). For technical reasons we consider not just the homotopy category of topological spaces, but rather the larger homotopy category of pairs.

A *topological pair* (X,A) consists of a topological space X and a subspace $A \subset X$. If A is empty, denoted by $\varnothing$, we shall not distinguish between the pair $(X,\varnothing)$ and the space X. A *subpair* $(X',A') \subset (X,A)$ consists of a pair with $X' \subset X$ and $A' \subset A$. A *map* $f\colon (X,A) \to (Y,B)$ between pairs is a continuous function f from X to Y such that $f(A) \subset B$, and as in Sec. 1.1, there is a category of topological pairs and maps between them which contains as full subcategories the category of topological spaces and continuous maps, as well as the category of pointed topological spaces and continuous maps.

Given a pair (X,A), we let $(X,A) \times I$ denote the pair $(X \times I, A \times I)$. Let $X' \subset X$ and suppose that $f_0, f_1\colon (X,A) \to (Y,B)$ agree on X' (that is, $f_0|X' = f_1|X'$). Then f_0 is *homotopic to* f_1 *relative to* X', denoted by $f_0 \simeq f_1$ rel X', if there exists a map

$$F\colon (X,A) \times I \to (Y,B)$$

such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for $x \in X$ and $F(x,t) = f_0(x)$ for $x \in X'$ and $t \in I$. Such a map F is called a *homotopy relative to* X' from f_0 to f_1 and is denoted by $F\colon f_0 \simeq f_1$ rel X'. If $X' = \varnothing$, we omit the phrase "relative to $\varnothing$." Clearly, $f_0 \simeq f_1$ rel X' implies $f_0 \simeq f_1$ rel X'' for any $X'' \subset X'$. A map from X to Y is said to be *null homotopic*, or *inessential*, if it is homotopic to some constant map.

For $t \in I$ define $h_t\colon (X,A) \to (X,A) \times I$ by $h_t(x) = (x,t)$. If $F\colon f_0 \simeq f_1$ rel X', then $Fh_0 = f_0$, $Fh_1 = f_1$, and $Fh_t|X' = f_0|X'$ for all $t \in I$. Therefore the collection $\{Fh_t\}_{t \in I}$ is a continuous one-parameter family of maps from (X,A) to (Y,B), agreeing on X', which connects $f_0 = Fh_0$ to $f_1 = Fh_1$[1]. Hence $f_0 \simeq f_1$ rel X' corresponds to the intuitive idea of continuously deforming f_0 into f_1 by maps all of which agree on X'. Note that if $f_0 \simeq f_1$ rel X' there will usually be many maps F which are homotopies relative to X' from f_0 to f_1 (see example 3 below).

1 EXAMPLE Let $X = Y = \mathbf{R}^n$ and define $f_0(x) = x$ and $f_1(x) = 0$ for $x \in \mathbf{R}^n$ (that is, $f_0 = 1_{\mathbf{R}^n}$ and f_1 is the constant map of $\mathbf{R}^n$ to its origin). If $F\colon \mathbf{R}^n \times I \to \mathbf{R}^n$ is defined by

$$F(x,t) = (1-t)x$$

then $F\colon f_0 \simeq f_1$ rel 0.

2 EXAMPLE Let $X = Y = I$ and define $f_0(t) = t$ and $f_1(t) = 0$ for $t \in I$. If $F\colon I \times I \to I$ is defined by

$$F(t,t') = (1-t')t$$

then $F\colon f_0 \simeq f_1$ rel 0.

3 EXAMPLE Let $X = Y = E^2 = \{z \in \mathbf{C} \mid z = re^{i\theta},\ 0 \le r \le 1\}$ and let $A = B = S^1 = \{z \in \mathbf{C} \mid z = e^{i\theta}\}$. Define $f_0\colon (E^2,S^1) \to (E^2,S^1)$ to be the identity map and $f_1\colon (E^2,S^1) \to (E^2,S^1)$ to be the reflection in the origin [that is, $f_1(re^{i\theta}) = re^{i(\theta+\pi)}$]. Define a homotopy $F\colon f_0 \simeq f_1$ rel 0 by $F(re^{i\theta},t) = re^{i(\theta+t\pi)}$. Another homotopy $F'\colon f_0 \simeq f_1$ rel 0 is defined by $F'(re^{i\theta},t) = re^{i(\theta-t\pi)}$.

[1] A one-parameter family $f_t\colon (X,A) \to (Y,B)$ for $t \in I$ is *continuous* if $f_t(x)$ is jointly continuous in t and x, in which case the function $(x,t) \to f_t(x)$ is a homotopy from f_0 to f_1. The corresponding function $t \to f_t$ from I to $(Y,B)^{(X,A)}$ is always continuous [where $(Y,B)^{(X,A)} = \{g\colon (X,A) \to (Y,B)\}$ topologized by the compact-open topology]. Conversely, in case X is a locally compact Hausdorff space, it follows from theorem 2.8 in the Introduction that for any continuous map $\varphi\colon I \to (Y,B)^{(X,A)}$ the one-parameter family $\varphi(t)$ is continuous and defines a homotopy from $\varphi(0)$ to $\varphi(1)$.

4 EXAMPLE Let X be an arbitrary space and let Y be a convex subset of $\mathbf{R}^n$. Let $f_0, f_1: X \to Y$ be maps which agree on some subspace $X' \subset X$. Then $f_0 \simeq f_1$ rel X', because the map $F: X \times I \to Y$ defined by

$$F(x,t) = tf_1(x) + (1 - t)f_0(x)$$

is a homotopy relative to X' from f_0 to f_1.

Example 4 is a generalization of examples 1 and 2. In example 3 the space E^2 is convex, but the homotopy between f_0 and f_1 cannot be taken to be a particular case of the homotopy in example 4, because it must keep S^1 mapped into itself at all stages, and S^1 is not convex.

To define the homotopy category we need the following easy results.

5 THEOREM *Homotopy relative to X' is an equivalence relation in the set of maps from (X,A) to (Y,B).*

PROOF *Reflexivity.* For $f: (X,A) \to (Y,B)$ define $F: f \simeq f$ rel X by $F(x,t) = f(x)$.

Symmetry. Given $F: f_0 \simeq f_1$ rel X', define $F': f_1 \simeq f_0$ rel X' by $F'(x,t) = F(x, 1 - t)$.

Transitivity. Given $F: f_0 \simeq f_1$ rel X' and $G: f_1 \simeq f_2$ rel X', define $H: f_0 \simeq f_2$ rel X' by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le ½ \\ G(x, 2t - 1) & ½ \le t \le 1 \end{cases}$$

Note that H is continuous because its restriction to each of the closed sets $X \times [0,½]$ and $X \times [½,1]$ is continuous. ∎

It follows that the set of maps from (X,A) to (Y,B) is partitioned into disjoint equivalence classes by the relation of homotopy relative to X'. These equivalence classes are called *homotopy classes relative to X'*. We use $[X,A; Y,B]_{X'}$ to denote this set of homotopy classes. Given $f: (X,A) \to (Y,B)$, we use $[f]_{X'}$ to denote the element of $[X,A; Y,B]_{X'}$ determined by f. Homotopy classes relative to the empty set will be denoted by omitting the subscript X'.

6 THEOREM *Composites of homotopic maps are homotopic.*

PROOF Let $f_0, f_1: (X,A) \to (Y,B)$ be homotopic relative to X' and let $g_0, g_1: (Y,B) \to (Z,C)$ be homotopic relative to Y', where $f_1(X') \subset Y'$. To show that $g_0f_0, g_1f_1: (X,A) \to (Z,C)$ are homotopic relative to X', let $F: f_0 \simeq f_1$ rel X' and $G: g_0 \simeq g_1$ rel Y'. Then the composite

$$(X,A) \times I \xrightarrow{F} (Y,B) \xrightarrow{g_0} (Z,C)$$

is a homotopy relative to X' from g_0f_0 to g_0f_1, and the composite

$$(X,A) \times I \xrightarrow{f_1 \times 1_I} (Y,B) \times I \xrightarrow{G} (Z,C)$$

is a homotopy relative to $f_1^{-1}(Y')$ from g_0f_1 to g_1f_1. Since $X' \subset f_1^{-1}(Y')$, we have shown that $g_0f_0 \simeq g_0f_1$ rel X' and $g_0f_1 \simeq g_1f_1$ rel X'. The result now follows from theorem 5. ∎

The last result shows that there is a *homotopy category of pairs* whose objects are topological pairs and whose morphisms are homotopy classes (relative to $\varnothing$). This category contains as full subcategories the *homotopy category of topological spaces* (also shortened to *homotopy category*) and the *homotopy category of pointed topological spaces.* There is a covariant functor from the category of pairs and maps to the homotopy category of pairs whose object function is the identity map and whose mapping function sends a map f to its homotopy class $[f]$. As pointed out at the beginning of the section, most of the algebraic functors we consider will be defined from the appropriate homotopy category. A diagram of topological pairs and maps is said to be *homotopy commutative* if it can be made a commutative diagram in the homotopy category (that is, when each map is replaced by its homotopy class).

As in example 1.2.4, for any pair (P,Q) there is a covariant functor $\pi_{(P,Q)}$ (or a contravariant functor $\pi^{(P,Q)}$) from the homotopy category of pairs to the category of sets and functions defined by $\pi_{(P,Q)}\ (X,A) = [P,Q;\ X,A]$ (or $\pi^{(P,Q)}\ (X,A) = [X,A;\ P,Q]$), and if $f\colon (X,A) \to (Y,B)$, then $\pi_{(P,Q)}\ ([f]) = f_\#$ (or $\pi^{(P,Q)}([f]) = f^\#$), where $f_\#[g] = [fg]$ for $g\colon (P,Q) \to (X,A)$ (or $f^\#[h] = [hf]$ for $h\colon (Y,B) \to (P,Q)$). If $\alpha\colon (P,Q) \to (P',Q')$, there is a natural transformation $\alpha^\#$ from $\pi_{(P',Q')}$ to $\pi_{(P,Q)}$ and a natural transformation $\alpha_\#$ from $\pi^{(P,Q)}$ to $\pi^{(P',Q')}$.

A map $f\colon (X,A) \to (Y,B)$ is called a *homotopy equivalence* if $[f]$ is an equivalence in the homotopy category of pairs. A map $g\colon (Y,B) \to (X,A)$ is called a *homotopy inverse* of f if $[g] = [f]^{-1}$ in the homotopy category. Pairs (X,A) and (Y,B) are said to have the *same homotopy type* if they are equivalent in the homotopy category.

The simplest nonempty space is a one-point space. We characterize the homotopy type of such a space as follows. A topological space X is said to be *contractible* if the identity map of X is homotopic to some constant map of X to itself. A homotopy from 1_X to the constant map of X to $x_0 \in X$ is called a *contraction* of X to x_0. Examples 1 and 2 show that $\mathbf{R}^n$ and I are contractible, and example 4 shows that any convex subset of $\mathbf{R}^n$ is contractible. The following lemma may be regarded as a generalization of the result of example 4.

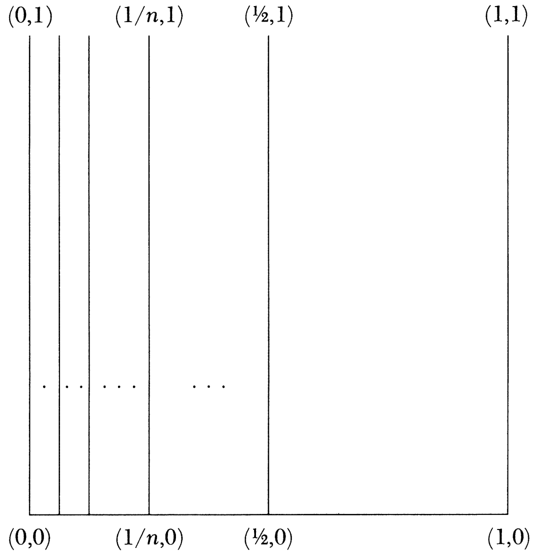
7 LEMMA *Any two maps of an arbitrary space to a contractible space are homotopic.*

PROOF Let Y be a contractible space and suppose $1_Y \simeq c$, where c is a constant map of Y to itself. Let $f_0, f_1\colon X \to Y$ be arbitrary. By theorem 6, $f_0 = 1_Y f_0 \simeq cf_0$, and similarly, $f_1 \simeq cf_1$. Since $cf_0 = cf_1$, it follows from theorem 5 that $f_0 \simeq f_1$. ■

8 COROLLARY *If Y is contractible, any two constant maps of Y to itself are homotopic, and the identity map is homotopic to any constant map of Y to itself.* ■

It is interesting to observe that lemma 7 cannot be strengthened to the case of relative homotopy. That is, if f_0 and f_1 are maps of X into a contractible space Y which agree on $X' \subset X$, it need not be true that $f_0 \simeq f_1$ rel X' (although example 4 shows this to be true for convex subsets of $\mathbf{R}^n$). The following example illustrates this and will be referred to again later.

9 EXAMPLE The *comb space* Y illustrated in the diagram



Comb space

is defined by

$$Y = \{(x,y) \in \mathbf{R}^2 \mid 0 \leq y \leq 1, x = 0, 1/n \quad \text{or} \quad y = 0, 0 \leq x \leq 1\}$$

Let F: $Y \times I \to Y$ be defined by $F((x,y), t) = (x, (1 - t)y)$. Then F is a homotopy from 1_Y to the projection of Y to the x axis. Since the latter map is homotopic to a constant map, Y is contractible. Let c: $Y \to Y$ be the constant map of Y to the point $(0,1)$. By corollary 8, $1_Y \simeq c$, but even though these two maps agree on $(0,1)$, there is no homotopy relative to $(0,1)$ between them.

The following theorem shows that contractible spaces are homotopically as simple as possible.

10 THEOREM *A space is contractible if and only if it has the same homotopy type as a one-point space.*

PROOF Assume that X is contractible and let F: $X \times I \to X$ be a contraction of X to a point $x_0 \in X$. Let P be the one-point space consisting of x_0 and let f: $X \to P$ and j: $P \subset X$. Then $fj = 1_P$ and F: $1_X \simeq jf$. Therefore $[j] = [f]^{-1}$, and f is a homotopy equivalence from X to P.

Conversely, if X has the same homotopy type as a one-point space P, let f: $X \to P$ be a homotopy equivalence with homotopy inverse g: $P \to X$. Then $1_X \simeq gf$. Because gf is a constant map, X is contractible. ∎

11 COROLLARY *Two contractible spaces have the same homotopy type, and any continuous map between contractible spaces is a homotopy equivalence.*

PROOF The first part follows from theorem 10 and the transitivity of the relation of having the same homotopy type. The second part follows from the first part and lemma 7 (and from the obvious fact that any map homotopic to a homotopy equivalence is itself a homotopy equivalence). ■

The next result establishes an important relation between homotopy and the extendability of maps.

12 THEOREM *Let p_0 be any point of S^n and let f: $S^n \to Y$. The following are equivalent:*

(a) f is null homotopic
(b) f can be continuously extended over E^{n+1}
(c) f is null homotopic relative to p_0

PROOF $(a) \Rightarrow (b)$. Let F: $f \simeq c$, where c is the constant map of S^n to $y_0 \in Y$. Define an extension f' of f over E^{n+1} by

$$f'(x) = \begin{cases} y_0 & 0 \le \|x\| \le \frac{1}{2} \\ F(x/\|x\|, 2 - 2\|x\|) & \frac{1}{2} \le \|x\| \le 1 \end{cases}$$

Since $F(x,1) = y_0$ for all $x \in S^n$, the map f' is well-defined. f' is continuous because its restriction to each of the closed sets $\{x \in E^{n+1} \mid 0 \le \|x\| \le \frac{1}{2}\}$ and $\{x \in E^{n+1} \mid \frac{1}{2} \le \|x\| \le 1\}$ is continuous. Since $F(x,0) = f(x)$ for $x \in S^n$, $f' \mid S^n = f$ and f' is a continuous extension of f to E^{n+1}.

$(b) \Rightarrow (c)$. If f has the continuous extension f': $E^{n+1} \to Y$, define F: $S^n \times I \to Y$ by

$$F(x,t) = f'((1 - t)x + tp_0)$$

Then $F(x,0) = f'(x) = f(x)$ and $F(x,1) = f'(p_0)$ for $x \in S^n$. Since $F(p_0,t) = f'(p_0)$ for $t \in I$, F is a homotopy relative to p_0 from f to the constant map to $f'(p_0)$.

$(c) \Rightarrow (a)$. This is obvious. ■

Combining theorem 12 with lemma 7, we obtain the following result.

13 COROLLARY *Any continuous map from S^n to a contractible space has a continuous extension over E^{n+1}.* ■

4 RETRACTION AND DEFORMATION

This section is concerned mainly with inclusion maps. We consider whether such a map has a left inverse, a right inverse, or a two-sided inverse in either the category of topological spaces and continuous maps or the homotopy category.[1]

[1] Many of the results in this section can be found in R. H. Fox, On homotopy type and deformation retracts, *Annals of Mathematics*, vol. 44, pp. 40–50, 1943 (see also H. Samelson, Remark on a paper by R. H. Fox, *Annals of Mathematics*, vol. 45, pp. 448–449, 1944).

A subspace A of X is called a *retract* of X if the inclusion map i: $A \subset X$ has a left inverse in the category of topological spaces and continuous maps. Hence A is a retract of X if and only if there is a continuous map r: $X \to A$ such that $ri = 1_A$ [that is, $r(x) = x$ for $x \in A$]. Such a map r is called a *retraction* of X to A.

A subspace A of X is called a *weak retract* of X if the inclusion map i: $A \subset X$ has a left homotopy inverse (that is, a left inverse in the homotopy category). Thus A is a weak retract of X if and only if there is a continuous map r: $X \to A$ such that $ri \simeq 1_A$. Such a map r is called a *weak retraction* of X to A.

Any one-point subspace is a retract of any larger space containing it. A discrete space with more than one point is never a weak retract of a connected space containing it. If A is a retract of X, it is a weak retract of X. The converse is not true, as is shown by the following example.

1 EXAMPLE Let X be the closed unit square I^2 in $\mathbf{R}^2$ and let $A \subset X$ be the comb space of example 1.3.9. Then A and X are both contractible, and by corollary 1.3.11, the inclusion map $A \subset X$ is a homotopy equivalence. Therefore A is a weak retract of X. However, it can be shown that A is not a retract of X.

Despite the fact that, in general, a weak retract need not be a retract, these concepts do coincide when A is a suitable subspace of X. This occurs frequently enough to warrant special consideration and will prove of use later. Let (X,A) be a pair and Y be a space. (X,A) is said to have the *homotopy extension property with respect to* Y if, given maps g: $X \to Y$ and G: $A \times I \to Y$ such that $g(x) = G(x,0)$ for $x \in A$, there is a map F: $X \times I \to Y$ such that $F(x,0) = g(x)$ for $x \in X$ and $F|A \times I = G$. If g is regarded as a map of $X \times 0$ to Y, the existence of F is equivalent to the existence of a map represented by the dotted arrow which makes the following diagram commutative:

$$\begin{tikzcd}
A \times 0 & \subset & A \times I \arrow[dl, "G"'] \\
\cap & Y & \cap \\
X \times 0 \arrow[ur, "g"] & \subset & X \times I \arrow[ul, dotted]
\end{tikzcd}$$

If (X,A) has the homotopy extension property with respect to Y and f_0, f_1: $A \to Y$ are homotopic, then if f_0 has an extension to X, so does f_1; for if g: $X \to Y$ is an extension of f_0 and G: $A \times I \to Y$ is a homotopy from f_0 to f_1, the homotopy extension property implies the existence of a map F: $X \times I \to Y$ which is an extension of G, therefore $F(x,1)$ is an extension of f_1. It follows that whether or not a map $A \to Y$ can be extended over X is a property of the homotopy class of that map. Therefore the homotopy extension property implies that the extension problem for maps $A \to Y$ is a problem in the homotopy category.

Of particular importance is the case when (X,A) has the homotopy extension property with respect to any space. More generally, a map $f: X' \to X$ is called a *cofibration* if, given maps $g: X \to Y$ and $G: X' \times I \to Y$ (where Y is arbitrary) such that $g(f(x')) = G(x',0)$ for $x' \in X'$, there is a map $F: X \times I \to Y$ such that $F(x,0) = g(x)$ for $x \in X$ and $F(f(x'), t) = G(x',t)$ for $x' \in X'$ and $t \in I$. If g is regarded as a map of $X \times 0$ to Y, the existence of F is equivalent to the existence of a map represented by the dotted arrow which makes the following diagram commutative:

$$\begin{tikzcd}
X' \times 0 \arrow[dd, "f \times 1_0"'] \arrow[rr, phantom, "\subset"] & & X' \times I \arrow[dl, "G"'] \arrow[dd, "f \times 1_I"] \\
& Y & \\
X \times 0 \arrow[ur, "g"] \arrow[rr, phantom, "\subset"] & & X \times I \arrow[ul, dashed]
\end{tikzcd}$$

Thus an inclusion map $i: A \subset X$ is a cofibration if and only if (X,A) has the homotopy extension property with respect to any space.

2 THEOREM *If (X,A) has the homotopy extension property with respect to A, then A is a weak retract of X if and only if A is a retract of X.*

PROOF We show that any weak retraction $r: X \to A$ is, in fact, homotopic to a retraction. Let $i: A \subset X$; then $ri \simeq 1_A$. Let $G: A \times I \to A$ be a homotopy from ri to 1_A; then $G(x,0) = r(x)$ for $x \in A$. Because (X,A) has the homotopy extension property with respect to A, there is a map $F: X \times I \to A$ which extends G such that $F(x,0) = r(x)$ for $x \in X$. If $r': X \to A$ is defined by $r'(x) = F(x,1)$, then r' is a retraction of X to A, and F is a homotopy from r to r'. ∎

We can just as well consider inclusion maps with right homotopy inverses as those with left homotopy inverses. This leads to the following definitions. Given $X' \subset X$, a *deformation* D of X' in X is a homotopy

$$D: X' \times I \to X$$

such that $D(x',0) = x'$ for $x' \in X'$. If, moreover, $D(X' \times 1)$ is contained in a subspace A of X, D is said to be a *deformation of X' into A* and X' is said to be *deformable in X into A*. A space X is said to be *deformable* into a subspace A if it is deformable in itself into A. Thus a space X is contractible if and only if it is deformable into one of its points.

3 LEMMA *A space X is deformable into a subspace A if and only if the inclusion map $i: A \subset X$ has a right homotopy inverse.*

PROOF If i has a right homotopy inverse $f: X \to A$, then $if \simeq 1_X$. Let $F: X \times I \to X$ be a homotopy from 1_X to if; then $F(x,0) = x$, so F is a deformation of X, and $F(X \times 1) = if(X) \subset A$, so X is deformable into A.

Conversely, if X is deformable into A, let $D: X \times I \to X$ be a deformation such that $D(X \times 1) \subset A$. Let $f: X \to A$ be defined by the equation

$$if(x) = D(x,1) \qquad x \in X$$

Then D: $1_X \simeq if$, showing that f is a right homotopy inverse of i. ∎

Note that an inclusion map i: $A \subset X$ never has a right inverse in the category of topological spaces and continuous maps except in the trivial case $A = X$.

We now consider inclusion maps which are homotopy equivalences. A subspace $A \subset X$ is called a *weak deformation retract* of X if the inclusion map i: $A \subset X$ is a homotopy equivalence. From lemma 1.1.1 and lemma 3 above we obtain the following result.

4 LEMMA *A is a weak deformation retract of X if and only if A is a weak retract of X and X is deformable into A.* ∎

As was the case with the concept of weak retract, there are more useful concepts than that of weak deformation retract. The subspace A is a *strong deformation retract* of X if there is a retraction r of X to A such that if i: $A \subset X$, then $1_X \simeq ir$ rel A. If F: $1_X \simeq ir$ rel A, F is called a *strong deformation retraction* of X to A.

There is an intermediate concept useful in comparing the weak and strong forms already defined. A subspace A is called a *deformation retract* of X if there is a retraction r of X to A such that if i: $A \subset X$, then $1_X \simeq ir$. If F: $1_X \simeq ir$, F is called a *deformation retraction* of X to A. A homotopy F: $X \times I \to X$ is a deformation retraction if and only if $F(x,0) = x$ for $x \in X$, $F(X \times 1) \subset A$, and $F(x,1) = x$ for $x \in A$. It is a strong deformation retraction if and only if it also satisfies the condition $F(x,t) = x$ for $x \in A$ and $t \in I$.

5 EXAMPLE It follows from example 1.3.4 that any one-point subset of a convex subset of $\mathbf{R}^n$ is a strong deformation retract of the convex set.

6 EXAMPLE S^n is a strong deformation retract of $\mathbf{R}^{n+1} - 0$. In fact the map F: $(\mathbf{R}^{n+1} - 0) \times I \to \mathbf{R}^{n+1} - 0$ defined by

$$F(x,t) = (1 - t)x + \frac{tx}{\|x\|} \qquad x \in \mathbf{R}^{n+1} - 0,\ t \in I$$

is a strong deformation retraction of $\mathbf{R}^{n+1} - 0$ to S^n.

It is clear that a strong deformation retract is a deformation retract, and a deformation retract is a weak deformation retract. The following examples show that neither of these implications is reversible.

7 EXAMPLE As in example 1 above, let X be the closed unit square and A be the comb space. As pointed out in example 1, the inclusion map $A \subset X$ is a homotopy equivalence, but A is not a retract of X. Therefore A is a weak deformation retract of X which is not a deformation retract of X.

8 EXAMPLE Let X be the comb space and A be the one-point subspace of X consisting of the point $(0,1)$. Because X is contractible, there is a homotopy F from 1_X to the constant map of X to A. Such a map F is a deformation retraction of X to A. However, as was remarked in example 1.3.9, there is no homotopy relative to A from 1_X to the constant map to A; therefore A is a deformation retract of X which is not a strong deformation retract of X.

In the presence of suitable homotopy extension properties the three concepts of deformation retract coincide, and we shall now prove this.

9 LEMMA *If X is deformable into a retract A, then A is a deformation retract of X.*

PROOF Let $r: X \to A$ be a retraction and let $i: A \subset X$. Then r is a left homotopy inverse of i. Because X is deformable into A, it follows from lemma 3 that i has a right homotopy inverse. By lemma 1.1.1, r is also a right homotopy inverse of i. Since $1_X \simeq ir$, A is a deformation retract of X. ■

Combining lemma 9 with theorem 2 yields the following corollary.

10 COROLLARY *If (X,A) has the homotopy extension property with respect to A, then A is a weak deformation retract of X if and only if A is a deformation retract of X.* ■

11 THEOREM *If $(X \times I, (X \times 0) \cup (A \times I) \cup (X \times 1))$ has the homotopy extension property with respect to X and A is closed in X, then A is a deformation retract of X if and only if A is a strong deformation retract of X.*

PROOF If A is a deformation retract of X, let $F: X \times I \to X$ be a homotopy from 1_X to ir, where $r: X \to A$ is a retraction and $i: A \subset X$. A homotopy

$$G: [(X \times 0) \cup (A \times I) \cup (X \times 1)] \times I \to X$$

is defined by the equations

$$\begin{array}{ll} G((x,0), t') = x & x \in X,\ t' \in I \\ G((x,t), t') = F(x, (1 - t')t) & x \in A;\ t, t' \in I \\ G((x,1), t') = F(r(x), 1 - t') & x \in X,\ t' \in I \end{array}$$

G is well-defined, because for $x \in A$

$$G((x,0), t') = x = F(x,0)$$

by the first two equations and

$$G((x,1), t') = F(x, 1 - t') = F(r(x), 1 - t')$$

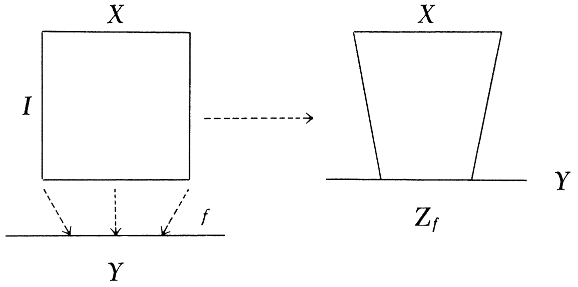
by the last two equations. G is continuous because its restriction to each of the closed sets $(X \times 0) \times I$, $(A \times I) \times I$, and $(X \times 1) \times I$ is continuous. For $(x,t) \in (X \times 0) \cup (A \times I) \cup (X \times 1)$, $G((x,t), 0) = F(x,t)$ [because $F(x,0) = x$, and since r is a retraction, $F(r(x), 1) = ir(r(x)) = r(x) = F(x,1)$]. Therefore G restricted to $[(X \times 0) \cup (A \times I) \cup (X \times 1)] \times 0$ can be extended to $(X \times I) \times 0$. From the homotopy extension property in the hypothesis, G restricted to $[(X \times 0) \cup (A \times I) \cup (X \times 1)] \times 1$ can be extended to $(X \times I) \times 1$. Let G': $(X \times I) \times 1 \to X$ be such an extension, and define $H: X \times I \to X$

by $H(x,t) = G'((x,t), 1)$. Then we have the equations

$$\begin{aligned} H(x,0) &= G'((x,0), 1) = G((x,0), 1) = x && x \in X \\ H(x,1) &= G((x,1), 1) = F(r(x),0) = r(x) && x \in X \\ H(x,t) &= G((x,t), 1) = F(x,0) = x && x \in A,\ t \in I \end{aligned}$$

Therefore H is a homotopy relative to A from 1_X to ir, so A is a strong deformation retract of X. ■

The next result asserts that any map is equivalent in the homotopy category to an inclusion map that is a cofibration. Let $f\colon X \to Y$ and let Z_f denote the quotient space obtained from the topological sum of $X \times I$ and Y by identifying $(x,1) \in X \times I$ with $f(x) \in Y$. Z_f is called the *mapping cylinder* of f and is depicted in the diagram



Mapping cylinder

We use $[x,t]$ to denote the point of Z_f corresponding to $(x,t) \in X \times I$ under the identification map and $[y]$ to denote the point of Z_f corresponding to $y \in Y$ (thus $[x,1] = [f(x)]$ for $x \in X$). There is an imbedding $i\colon X \to Z_f$ with $i(x) = [x,0]$ and an imbedding $j\colon Y \to Z_f$ with $j(y) = [y]$. X and Y are regarded as subspaces of Z_f by means of these imbeddings. A retraction $r\colon Z_f \to Y$ is defined by $r[x,t] = [f(x)]$ for $x \in X$ and $t \in I$ and $r[y] = [y]$ for $y \in Y$.

12 THEOREM *Given a map $f\colon X \to Y$, there is a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & Z_f \arrow[dl, "r"] \\ & Y & \end{tikzcd}$$

such that (*a*) $1_{Z_f} \simeq jr$ rel Y (*b*) *i is a cofibration*

PROOF By definition, $ri = f$, and the triangle is commutative.

(*a*) A homotopy $F\colon Z_f \times I \to Z_f$ is defined by

$$\begin{aligned} F([x,t], t') &= [x, (1-t')t + t'] && x \in X;\ t, t' \in I \\ F([y],t') &= [y] && y \in Y,\ t' \in I \end{aligned}$$

Then $F\colon 1_{Z_f} \simeq jr$ rel Y.
(F is continuous because $Z_f \times I$ has the topology coinduced by the maps $X \times I \times I \to Z_f \times I$ sending (x,t,t') to $([x,t],t')$ and $Y \times I \to Z_f \times I$ sending (y,t') to $([y],t')$.)

(*b*) Let $g\colon Z_f \to W$ and $G\colon X \times I \to W$ be such that $g([x,0]) = G(x,0)$ for $x \in X$. If $H\colon Z_f \times I \to W$ is defined by the equations

$$H([y],t') = g[y] \qquad y \in Y,\ t' \in I$$

$$H([x,t],\ t') = \begin{cases} g[x,\ (2t - t')/(2 - t')] & 0 \le t' \le 2t \le 2,\ x \in X \\ G(x,\ (t' - 2t)/(1 - t)) & 0 \le 2t \le t' \le 1,\ x \in X \end{cases}$$

then $H([x,t],\ 0) = g[x,t]$ and $H([y],0) = g[y]$, and $H|X \times I = G$. ∎

It follows that the map $i\colon X \subset Z_f$ is a cofibration equivalent in the homotopy category to the map $f\colon X \to Y$. The mapping cylinder can be used to prove the following amusing result.

13 THEOREM *Two spaces X and Y have the same homotopy type if and only if they can be imbedded as weak deformation retracts of the same space Z.*

PROOF If X and Y can be imbedded as weak deformation retracts of the same space Z, then X and Y each have the same homotopy type as Z. Therefore X and Y have the same homotopy type.

Conversely, if $f\colon X \to Y$ is a homotopy equivalence, it follows from theorem 12 that if Z_f is the mapping cylinder of f, then the composite $X \xrightarrow{i} Z_f \xrightarrow{r} Y$ is a homotopy equivalence. Because r is a homotopy equivalence, this implies that i is a homotopy equivalence. By theorem 12*a*, $j\colon Y \to Z_f$ is a homotopy equivalence. Therefore X and Y are imbedded as weak deformation retracts in Z_f. ∎

All the foregoing concepts can also be considered for pairs. For example, a pair $(X',A') \subset (X,A)$ is a *strong deformation retract* if there is a map $F\colon (X,A) \times I \to (X,A)$ such that $F(x,0) = x$ for $x \in X$, $F(X \times 1) \subset X'$, $F(A \times 1) \subset A'$, and $F(x',t) = x'$ for $x' \in X'$ and $t \in I$. The *mapping cylinder* of a map $f\colon (X,A) \to (Y,B)$, where A is closed in X, is the pair (Z_{f_1},Z_{f_2}), where Z_{f_1} is the mapping cylinder of the map $f_1\colon X \to Y$ defined by f and Z_{f_2} is the mapping cylinder of the map $f_2\colon A \to B$ defined by f. A map $f\colon (X',A') \to (X,A)$ is a *cofibration* if, given maps $g\colon (X,A) \to (Y,B)$ and $G\colon (X',A') \times I \to (Y,B)$ [where (Y,B) is arbitrary] such that $G(x',0) = gf(x')$ for $x' \in X'$, there exists a map $F\colon (X,A) \times I \to (Y,B)$ such that $F(x,0) = g(x)$ for $x \in X$ and $G(x',t) = F(f(x'),\ t)$ for $x' \in X'$ and $t \in I$. All the results remain valid when suitably formulated for pairs.

5 *H* SPACES

In some cases it is possible to introduce a natural group structure in the set of homotopy classes of maps from one space (or pair) to another. In this section we consider spaces P such that $[X;P]$ admits a group structure for all X. It is not surprising that there is a close relation between natural group structures on $[X;P]$ for all X and "grouplike" structures on P.

We shall work in the homotopy category of pointed topological spaces, although much of what we do is also valid in the homotopy category of topological spaces. If X and Y are pointed topological spaces, $[X;Y]$ will denote the set of base-point-preserving homotopy classes of continuous maps $X \to Y$ (with all homotopies understood to be relative to the base point). Thus $[X;Y]$ is the set of morphisms from X to Y in the homotopy category of pointed topological spaces.

One method of obtaining a group structure on $[X;P]$ is to start with a group structure on P. Thus, let P be a topological group with identity element as base point. There is a law of composition in the set of all base-point-preserving continuous maps from X to P defined by pointwise multiplication of functions. That is, if g_1, g_2: $X \to P$, then g_1g_2: $X \to P$ is defined by $g_1g_2(x) = g_1(x)g_2(x)$, where the right-hand side is the group product in P. With this law of composition, the set of base-point-preserving continuous maps from X to P is a group (which is abelian if P is abelian). The law of composition carries over to give an operation on homotopy classes such that $[g_1][g_2] = [g_1g_2]$, and we have the following theorem.

1 THEOREM *If P is a topological group, π^P is a contravariant functor from the homotopy category of pointed topological spaces to the category of groups and homomorphisms.* ∎

We give two examples.

2 S^1 is an abelian topological group (the multiplicative group of complex numbers of norm 1). Therefore $[X;S^1]$ is an abelian group, and if f: $X \to Y$, then $f^\#$: $[Y;S^1] \to [X;S^1]$ is a homomorphism.

3 S^3 is a topological group (the multiplicative group of quaternions of norm 1). Therefore $[X;S^3]$ is a group, and if f: $X \to Y$, then $f^\#$: $[Y;S^3] \to [X;S^3]$ is a homomorphism.

This group structure on $[X;P]$ was deduced from a group structure on the set of base-point-preserving continuous maps from X to P. There are situations in which $[X;P]$ admits a natural group structure, but the set of base-point-preserving continuous maps from X to P has no group structure. For example, if P is a pointed space having the same homotopy type as some topological group P', then π^P is naturally equivalent to $\pi^{P'}$. Therefore π^P can be regarded as a functor to the category of groups. The following definitions will be used to describe the additional structure needed on a pointed space P in order that π^P take values in the category of groups and homomorphisms.

If f: $X \to Y$ and g: $X \to Z$, we define

$$(f,g)\colon X \to Y \times Z$$

to be the map $(f,g)(x) = (f(x),g(x))$ for $x \in X$.

An *H space* consists of a pointed topological space P together with a continuous multiplication

$$\mu\colon P \times P \to P$$

for which the (unique) constant map $c\colon P \to P$ is a *homotopy identity*, that is, each composite

$$P \xrightarrow{(c,1)} P \times P \xrightarrow{\mu} P \qquad \text{and} \qquad P \xrightarrow{(1,c)} P \times P \xrightarrow{\mu} P$$

is homotopic to 1_P. The multiplication μ is said to be *homotopy associative* if the square

$$\begin{array}{ccc} P \times P \times P & \xrightarrow{\mu \times 1} & P \times P \\ {\scriptstyle 1 \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ P \times P & \xrightarrow{\mu} & P \end{array}$$

is homotopy commutative, that is, $\mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu)$. A continuous function $\varphi\colon P \to P$ is called a *homotopy inverse* for P and μ if each of the composites

$$P \xrightarrow{(1,\varphi)} P \times P \xrightarrow{\mu} P \qquad \text{and} \qquad P \xrightarrow{(\varphi,1)} P \times P \xrightarrow{\mu} P$$

is homotopic to $c\colon P \to P$.

A homotopy-associative H space with a homotopy inverse satisfies the group axioms up to homotopy. Such a pointed space is called an *H group*. Clearly, any topological group is an H group.

A multiplication μ in an H space is said to be *homotopy abelian* if the triangle

$$\begin{array}{ccc} P \times P & \xrightarrow{T} & P \times P \\ & {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & P & \end{array}$$

where $T(p_1,p_2) = (p_2,p_1)$, is homotopy commutative. An H group with homotopy-abelian multiplication is called an *abelian H* group.

If P and P' are H spaces with multiplications μ and μ', respectively, a continuous map $\alpha\colon P \to P'$ is called a *homomorphism* if the square

$$\begin{array}{ccc} P \times P & \xrightarrow{\mu} & P \\ {\scriptstyle \alpha \times \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ P' \times P' & \xrightarrow{\mu'} & P' \end{array}$$

is homotopy commutative.

4 THEOREM *A pointed space having the same homotopy type as an H space (or an H group) is itself an H space (or H group) in such a way that the homotopy equivalence is a homomorphism.*

PROOF Let $f\colon P \to P'$ and $g\colon P' \to P$ be homotopy inverses and let P be an H space with multiplication $\mu\colon P \times P \to P$. If $\mu'\colon P' \times P' \to P'$ is defined to be the composite

$$P' \times P' \xrightarrow{g \times g} P \times P \xrightarrow{\mu} P \xrightarrow{f} P'$$

then μ' is a continuous multiplication in P' and the composite $P' \xrightarrow{(1,c')} P' \times P' \xrightarrow{\mu'} P'$ equals the composite $P' \xrightarrow{g} P \xrightarrow{(1,c)} P \times P \xrightarrow{\mu} P \xrightarrow{f} P'$, which is homotopic to the composite $P' \xrightarrow{g} P \xrightarrow{f} P'$. Because $fg \simeq 1_{P'}$, the map $\mu' \circ (1,c')$ is homotopic to $1_{P'}$. Similarly, the map $\mu' \circ (c',1)$ is homotopic to $1_{P'}$. Therefore P' is an H space. Because the square

$$\begin{array}{ccc} P' \times P' & \xrightarrow{\mu'} & P' \\ {\scriptstyle g \times g}\downarrow & & \downarrow{\scriptstyle g} \\ P \times P & \xrightarrow{\mu} & P \end{array}$$

is homotopy commutative, g is a homomorphism (and so is f). If μ is homotopy associative or homotopy abelian, so is μ', and if $\varphi\colon P \to P$ is a homotopy inverse for P, then $f\varphi g\colon P' \to P'$ is a homotopy inverse for P'. ∎

Given an H space P, for any pointed space X there is a law of composition in $[X;P]$ defined by $[g_1][g_2] = [\mu \circ (g_1,g_2)]$. If P is an H group, $[X;P]$ becomes a group with this law of composition, and if $f\colon X \to Y$, then $f^\#\colon [Y,P] \to [X;P]$ is a homomorphism. Therefore we have the following theorem.

5 THEOREM *If P is an H group, π^P is a contravariant functor from the homotopy category of pointed topological spaces with values in the category of groups and homomorphisms. If P is an abelian H group, this functor takes values in the category of abelian groups.* ∎

It is interesting that the following converse of theorem 5 is also valid.

6 THEOREM *If P is a pointed space such that π^P takes values in the category of groups, then P is an H group (abelian if π^P takes values in the category of abelian groups). Furthermore, for any pointed space X, the group structure on $\pi^P(X)$ is the same as that given by theorem 5.*

PROOF Let $p_1\colon P \times P \to P$ and $p_2\colon P \times P \to P$ be the projections, and let $\mu\colon P \times P \to P$ be a map such that $[\mu] = [p_1] * [p_2]$, where $*$ is the law of composition in the group $[P \times P;\, P]$. For any maps $f, g\colon X \to P$, $(f,g)^\#\colon [P \times P;\, P] \to [X;P]$ is a homomorphism and

$$\begin{aligned}[\mu \circ (f,g)] &= (f,g)^\#[\mu] = (f,g)^\#([p_1] * [p_2]) \\ &= (f,g)^\#[p_1] * (f,g)^\#[p_2] = [f] * [g]\end{aligned}$$

This shows that the multiplication in $[X;P]$ is induced by the multiplication map μ.

Let X be a one-point space. The unique map $X \to P$ represents the identity element of the group $[X;P]$. Because the unique map $P \to X$ induces a homomorphism $[X;P] \to [P;P]$, it follows that the composite $P \to X \to P$, which is the constant map $c: P \to P$, represents the identity element of $[P;P]$. It follows that $\mu \circ (1_P,c) \simeq 1_P$ and $\mu \circ (c,1_P) \simeq 1_P$. Therefore P is an H space.

To prove that μ is homotopy associative, let $q_1, q_2, q_3: P \times P \times P \to P$ be the projections. Then

$$\begin{aligned}[\mu \circ (1 \times \mu)] &= (1 \times \mu)^\#[\mu] = (1 \times \mu)^\#[p_1] * (1 \times \mu)^\#[p_2] \\ &= [q_1] * [\mu(q_2,q_3)] = [q_1] * ([q_2] * [q_3])\end{aligned}$$

Similarly,

$$[\mu \circ (\mu \times 1)] = ([q_1] * [q_2]) * [q_3]$$

Because $[P \times P \times P; P]$ has an associative multiplication, $\mu \circ (1 \times \mu) \simeq \mu \circ (\mu \times 1)$.

To show that P has a homotopy inverse, let $\varphi: P \to P$ be such that $[1_P] * [\varphi] = [c]$; then $\mu(1_P,\varphi) \simeq c$. Also, $[\varphi] * [1_P] = [c]$, and so $\mu(\varphi,1_P) \simeq c$. Therefore φ is a homotopy inverse for P.

This proves that P is an H group and that the multiplication in π^P is induced from that on P. If $[P \times P; P]$ is an abelian group, a similar argument shows that P is an abelian H group. ∎

The following complement to theorems 5 and 6 is easily established by similar methods.

7 THEOREM *Let $\alpha: P \to P'$ be a map between H groups. Then $\alpha_\#$ is a natural transformation from π^P to $\pi^{P'}$ in the category of groups if and only if α is a homomorphism.* ∎

We describe a particularly useful example of an H group. Let Y be a pointed topological space with base point y_0. The *loop space* of Y (*based at* y_0), denoted by ΩY [or by $\Omega(Y,y_0)$], is defined to be the space of continuous functions $\omega: (I,\dot I) \to (Y,y_0)$ topologized by the compact-open topology. ΩY is regarded as a pointed space with base point ω_0 equal to the constant map of I to y_0. There is a map

$$\mu: \Omega Y \times \Omega Y \to \Omega Y$$

defined by

$$\mu(\omega,\omega')(t) = \begin{cases}\omega(2t) & 0 \le t \le 1/2 \\ \omega'(2t-1) & 1/2 \le t \le 1\end{cases}$$

To prove that μ is continuous, let $E: \Omega Y \times I \to Y$ be the evaluation map. By theorem 2.8 in the Introduction, it suffices to show that the composite

$$\Omega Y \times \Omega Y \times I \xrightarrow{\mu \times 1} \Omega Y \times I \xrightarrow{E} Y$$

is continuous. The formula which defines μ shows that this composite is continuous on each of the closed sets $\Omega Y \times \Omega Y \times [0,1/2]$ and $\Omega Y \times \Omega Y \times [1/2,1]$.

We construct a number of homotopies to show that ΩY is an H group.

Similar formulas will be used again in Sec. 1.7 to define homotopies of (nonclosed) paths in a topological space.

To prove that the map $\omega \to \mu(\omega,\omega_0)$ is homotopic to the identity map of ΩY, define $F\colon \Omega Y \times I \to \Omega Y$ by

$$F(\omega,t)(t') = \begin{cases} \omega\left(\dfrac{2t'}{t+1}\right) & 0 \le t' \le \dfrac{t+1}{2} \\ y_0 & \dfrac{t+1}{2} \le t' \le 1 \end{cases}$$

This formula shows that $E(F \times 1)\colon (\Omega Y \times I) \times I \to Y$ is continuous; therefore F is continuous and is a homotopy from the map $\omega \to \mu(\omega,\omega_0)$ to $1_{\Omega Y}$. Similarly, the map $\omega \to \mu(\omega_0,\omega)$ is homotopic to $1_{\Omega Y}$. Therefore ΩY is an H space with multiplication μ.

To show that μ is homotopy associative, define

$$G\colon \Omega Y \times \Omega Y \times \Omega Y \times I \to \Omega Y$$

by the formula

$$E(G \times 1)(\omega,\omega',\omega'',t,t') = \begin{cases} \omega\left(\dfrac{4t'}{t+1}\right) & 0 \le t' \le \dfrac{t+1}{4} \\ \omega'(4t' - t - 1) & \dfrac{t+1}{4} \le t' \le \dfrac{t+2}{4} \\ \omega''\left(\dfrac{4t' - 2 - t}{2 - t}\right) & \dfrac{t+2}{4} \le t' \le 1 \end{cases}$$

Then $G\colon \mu \circ (\mu \times 1_{\Omega Y}) \simeq \mu \circ (1_{\Omega Y} \times \mu)$, showing that μ is homotopy associative.

We define a homotopy inverse $\varphi\colon \Omega Y \to \Omega Y$ by $\varphi(\omega)(t) = \omega(1 - t)$. Then we define $H\colon \Omega Y \times I \to \Omega Y$ by

$$E(H \times 1)(\omega,t,t') = \begin{cases} y_0 & 0 \le t' \le \dfrac{t}{2} \\ \omega(2t' - t) & \dfrac{t}{2} \le t' \le \dfrac{1}{2} \\ \omega(2 - 2t' - t) & \dfrac{1}{2} \le t' \le 1 - \dfrac{t}{2} \\ y_0 & 1 - \dfrac{t}{2} \le t' \le 1 \end{cases}$$

H is a homotopy from the map $\omega \to \mu(\omega,\varphi(\omega))$ to the constant map of ΩY to itself. Similarly, there is a homotopy from the map $\omega \to \mu(\varphi(\omega),\omega)$ to the constant map of ΩY. Therefore φ is a homotopy inverse for ΩY, and ΩY is an H group.

If $h\colon Y \to Y'$ preserves base points, there is a map

$$\Omega h\colon \Omega Y \to \Omega Y'$$

defined by $\Omega h(\omega)(t) = h(\omega(t))$. Clearly, Ωh is a homomorphism, and we summarize these remarks about loop spaces as follows.

8 THEOREM *The loop functor Ω is a covariant functor from the category of pointed topological spaces and continuous maps to the category of H groups and continuous homomorphisms.* ∎

The functor Ω also preserves homotopies. That is, if $h_0, h_1\colon Y \to Y'$ are homotopic by a homotopy h_t, then $\Omega h_0, \Omega h_1\colon \Omega Y \to \Omega Y'$ are homotopic by a homotopy Ωh_t, which is a continuous homomorphism for each $t \in I$.

6 SUSPENSION

This section deals primarily with results dual to those of Sec. 1.5. We consider pointed spaces Q such that π_Q is a covariant functor from the homotopy category of pointed spaces to the category of groups and homomorphisms, and this leads to the concept of H cogroup, dual to that of H group. An important example of an H cogroup is the suspension of a pointed space, a concept dual to that of the loop space. The homotopy groups of a space defined in the section are examples of groups of homotopy classes of maps from suspensions to the space.

If X and Y are pointed topological spaces, their sum in the category of pointed topological spaces will be denoted by $X \vee Y$. If X has base point x_0 and Y has base point y_0, $X \vee Y$ may be regarded as the subspace $X \times y_0 \cup x_0 \times Y$ of $X \times Y$. If $f\colon X \to Z$ and $g\colon Y \to Z$, we let $(f,g)\colon X \vee Y \to Z$ be the map defined by the characteristic property of the sum [that is, $(f,g) \mid X = f$ and $(f,g) \mid Y = g$].

An *H cogroup* consists of a pointed topological space Q together with a continuous comultiplication

$$\nu\colon Q \to Q \vee Q$$

such that the following properties hold:

Existence of homotopy identity. If $c\colon Q \to Q$ is the (unique) constant map, each composite

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(c,1)} Q \qquad \text{and} \qquad Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(1,c)} Q$$

is homotopic to 1_Q.

Homotopy associativity. The square

$$\begin{array}{ccc} Q & \xrightarrow{\nu} & Q \vee Q \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle 1 \vee \nu} \\ Q \vee Q & \xrightarrow{\nu \vee 1} & Q \vee Q \vee Q \end{array}$$

is homotopy commutative.

Existence of homotopy inverse. There exists a map $\psi\colon Q \to Q$ such that each composite

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(1,\psi)} Q \qquad \text{and} \qquad Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(\psi,1)} Q$$

is homotopic to $c\colon Q \to Q$.

If X is any pointed space and Q is an H cogroup, there is a law of composition in $[Q;X]$ defined by $[f_1][f_2] = [(f_1,f_2) \circ \nu]$ which makes $[Q;X]$ a group.

An H cogroup is said to be *abelian* if the triangle

$$\begin{tikzcd} & Q \arrow[dl, "\nu"'] \arrow[dr, "\nu"] & \\ Q \vee Q \arrow[rr, "T'"] & & Q \vee Q \end{tikzcd}$$

where $T'(q_1,q_2) = (q_2,q_1)$ for $q_1, q_2 \in Q$, is homotopy commutative.

If Q and Q' are H cogroups with comultiplications ν and ν', respectively, a continuous map $\beta\colon Q \to Q'$ is called a *homomorphism* if the square

$$\begin{tikzcd} Q \arrow[r, "\nu"] \arrow[d, "\beta"'] & Q \vee Q \arrow[d, "\beta \vee \beta"] \\ Q' \arrow[r, "\nu'"] & Q' \vee Q' \end{tikzcd}$$

is homotopy commutative.

The proofs of the following theorems are dual to the proofs of the corresponding statements about H groups (see theorems 1.5.4, 1.5.5, 1.5.6, and 1.5.7) and are omitted.

1 THEOREM *A pointed space having the same homotopy type as an H cogroup is itself an H cogroup in such a way that the homotopy equivalence is a homomorphism.* ■

2 THEOREM *If Q is an H cogroup, π_Q is a covariant functor from the homotopy category of pointed spaces with values in the category of groups and homomorphisms. If Q is an abelian H cogroup, this functor takes values in the category of abelian groups.* ■

3 THEOREM *If Q is a pointed space such that π_Q takes values in the category of groups, then Q is an H cogroup (abelian if π_Q takes values in the category of abelian groups). Furthermore, the group structure on $\pi_Q(X)$ is identical with that determined by the H cogroup structure of Q as in theorem 2.* ■

4 THEOREM *If β: $Q \to Q'$ is a map between H cogroups, then $\beta\#$ is a natural transformation from $\pi_{Q'}$ to π_Q in the category of groups if and only if β is a homomorphism.* ■

We describe an example of an H cogroup dual to the loop-space example of an H group. Let Z be a pointed topological space with base point z_0. The *suspension* of Z, denoted by SZ, is defined to be the quotient space of $Z \times I$ in which $(Z \times 0) \cup (z_0 \times I) \cup (Z \times 1)$ has been identified to a single point. This is sometimes called the *reduced suspension* in the literature, the term "suspension" being used for the suspension in the category of spaces (no base points). The latter is defined to be the quotient space of $Z \times I$ in which $Z \times 0$ is identified to one point and $Z \times 1$ is identified to another point.

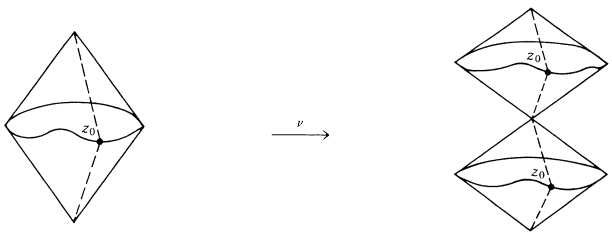
If $(z,t) \in Z \times I$, we use $[z,t]$ to denote the corresponding point of SZ under the quotient map $Z \times I \to SZ$. Then $[z,0] = [z_0,t] = [z',1]$ for all z, $z' \in Z$ and $t \in I$. The point $[z_0,0] \in SZ$ is also denoted by z_0, and SZ is a pointed space with base point z_0. If $f\colon Z \to Z'$, then $Sf\colon SZ \to SZ'$ is defined by $Sf[z,t] = [f(z), t]$. Thus S is a covariant functor from the category of pointed spaces and continuous maps. To show that it is a covariant functor to the category of H cogroups and homomorphisms, we define a comultiplication

$$\nu\colon SZ \to SZ \vee SZ$$

by the formula

$$\nu([z,t]) = \begin{cases} ([z,2t], z_0) & 0 \le t \le \tfrac{1}{2} \\ (z_0, [z, 2t-1]) & \tfrac{1}{2} \le t \le 1 \end{cases}$$

and illustrate it in the diagram (where the dotted lines are collapsed to one point).



SZ $\qquad\qquad$ SZ $\vee$ SZ

The map ν provides SZ with the structure of an H cogroup such that if $f\colon Z \to Z'$, then Sf is a homomorphism. This can be verified directly or deduced from properties of loop spaces already established. We follow the latter course.

The functors Ω and S defined from the category of pointed spaces and continuous maps to itself are examples of *adjoint functors.* This means that for pointed spaces Z and Y there is a natural equivalence

$$\hom(SZ,Y) \approx \hom(Z,\Omega Y)$$

where both sides are interpreted as the set of morphisms in the category of pointed spaces and continuous maps. This equivalence results from theorem 2.8 in the Introduction, and if $g\colon Z \to \Omega Y$, the corresponding $g'\colon SZ \to Y$ is defined by $g'[z,t] = g(z)(t)$ for $z \in Z$ and $t \in I$. It is obvious that if $h\colon Y \to Y'$, then $(\Omega h \circ g)' = h \circ g'$, and if $f\colon Z' \to Z$, then $(g \circ f)' = g' \circ Sf$. Therefore the equivalence $g \leftrightarrow g'$ comes from a natural equivalence from the functor $\hom(S\ \cdot\ ,\ \cdot\ )$ to the functor $\hom(\ \cdot\ ,\ \Omega\ \cdot\ )$.

This natural equivalence passes to morphisms in the homotopy category of pointed spaces. For pointed spaces a homotopy $G\colon Z \times I \to Y$ must map $z_0 \times I$ into y_0. Therefore it defines a map $F\colon Z \times I/z_0 \times I \to Y$. Because $S(Z \times I/z_0 \times I)$ can be identified with $SZ \times I/z_0 \times I$ by the homeomorphism

$$[(z,t), t'] \leftrightarrow ([z,t'], t) \qquad z \in Z;\ t, t' \in I$$

it follows that homotopies $F\colon Z \times I/z_0 \times I \to \Omega Y$ correspond bijectively to homotopies $F'\colon SZ \times I/z_0 \times I \to Y$. Therefore the equivalence above gives rise to an equivalence

$$[SZ;Y] \approx [Z;\Omega Y]$$

such that if the maps $g\colon Z \to \Omega Y$ and $g'\colon SZ \to Y$ are related by $g'[z,t] = g(z)(t)$, then $[g']$ corresponds to $[g]$. Hence there is a natural equivalence from the functor $[S\ \cdot\ ;\ \cdot\ ]$ to the functor $[\ \cdot\ ;\ \Omega\ \cdot\ ]$.

It follows from these remarks that for a fixed pointed space Z the functor π_{SZ} is naturally equivalent to the composite functor $\pi_Z \circ \Omega$. Here Ω is regarded as a covariant functor to the homotopy category of H groups and homomorphisms. Then the composite $\pi_Z \circ \Omega$ takes values in the category of groups and homomorphisms. By theorem 3, SZ is an H cogroup, and the map $\nu\colon SZ \to SZ \vee SZ$ defined above is the one which is the comultiplication in the H cogroup SZ (or is homotopic to it). In similar fashion, if $f\colon Z \to Z'$, the natural transformation $(Sf)_\#$ from $\pi_{SZ'}$ to π_{SZ} corresponds to the natural transformation $f_\#$ from the composite $\pi_{Z'} \circ \Omega$ to the composite $\pi_Z \circ \Omega$. Because the latter is a natural transformation in the category of groups, so is $(Sf)_\#$, and by theorem 4, Sf is a homomorphism of the H cogroup SZ to the H cogroup SZ'.

These statements are summarized as follows.

5 THEOREM *The suspension functor S is a covariant functor from the category of pointed spaces and maps to the category of H cogroups and continuous homomorphisms.* ■

The functor S also preserves homotopies. That is, if $f_0, f_1\colon Z \to Z'$ are homotopic by a homotopy f_t, then Sf_0, Sf_1 are homotopic by a homotopy Sf_t, which is a continuous homomorphism for each $t \in I$.

We now show that for $n \geq 1$ the sphere S^n is homeomorphic to a suspension, and thus obtain an interesting family of H cogroups. The corresponding functors are known as the homotopy group functors and are particularly important.

6 LEMMA *For $n \geq 0$, $S(S^n)$ is homeomorphic to S^{n+1}.*

PROOF Let $p_0 = (1,0, \ldots ,0)$ be the base point of S^n. We regard $\mathbf{R}^{n+1}$ as imbedded in $\mathbf{R}^{n+2}$ as the set of points in $\mathbf{R}^{n+2}$ whose $(n+2)$nd coordinate is 0. Then S^n is imbedded as an equator in S^{n+1}.

$$S^n = \{z \in \mathbf{R}^{n+2} \mid \|z\| = 1 \quad \text{and} \quad z_{n+2} = 0\}$$

and E^{n+1} is also imbedded in E^{n+2}:

$$E^{n+1} = \{z \in \mathbf{R}^{n+2} \mid \|z\| \leq 1 \quad \text{and} \quad z_{n+2} = 0\}$$

Let H_+ and H_- be the two closed hemispheres of S^{n+1} defined by the equator S^n. Then

$$H_+ = \{z \in S^{n+1} \mid z_{n+2} \geq 0\} \quad \text{and} \quad H_- = \{z \in S^{n+1} \mid z_{n+2} \leq 0\}$$

and $S^{n+1} = H_+ \cup H_-$ and $S^n = H_+ \cap H_-$. Furthermore, the projection map $\mathbf{R}^{n+2} \to \mathbf{R}^{n+1}$ defines projection maps p_+: $H_+ \to E^{n+1}$ and p_-: $H_- \to E^{n+1}$, which are homeomorphisms. A map f: $S(S^n) \to S^{n+1}$ is defined by

$$f[z,t] = \begin{cases} p_-^{-1}(2tz + (1 - 2t)p_0) & 0 \leq t \leq \frac{1}{2} \\ p_+^{-1}((2 - 2t)z + (2t - 1)p_0) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and is verified to be a homeomorphism f: $S(S^n) \approx S^{n+1}$. ■

For $n \geq 1$ the nth *homotopy group* functor π_n is the covariant functor on the homotopy category of pointed spaces defined by $\pi_n = \pi_{S^n}$. It follows from theorems 6 and 5 that these functors take values in the category of groups and homomorphisms.

In the last two sections of this chapter we give another definition of π_1 and study it in more detail. In Chapter 7 we return to the study of the higher homotopy groups π_n.

The following necessary and sufficient condition for a map $S^n \to X$ to represent the trivial element of $\pi_n(X)$ is an immediate consequence of theorem 1.3.12.

7 THEOREM *A map α: $S^n \to X$ represents the trivial element of $\pi_n(X)$ for $n \geq 1$ if and only if α can be continuously extended over E^{n+1}.* ■

Before leaving this section let us consider the interplay between two possible group structures on the set $[X;Y]$ for particular pointed spaces X and Y (for example, if X is an H cogroup and Y is an H group, this set can be given a group structure in two ways). It is a fact that under rather general circumstances two laws of composition on hom (X,Y) in a category are equal, and we establish this result.

8 THEOREM *Let X and Y be objects in a category and let $*$ and $*'$ be two laws of composition in* hom (X,Y) *such that*

(a) $*$ and $*'$ have a common two-sided identity element
(b) $*$ and $*'$ are mutually distributive

Then $*$ and $*'$ are equal, and each is commutative and associative.

PROOF Statement (a) means there is a map f_0: $X \to Y$ such that for any f: $X \to Y$

$$f * f_0 = f_0 * f = f = f *' f_0 = f_0 *' f$$

Statement (b) means that for f_1, f_2, g_1, g_2: $X \to Y$

$$(f_1 * f_2) *' (g_1 * g_2) = (f_1 *' g_1) * (f_2 *' g_2)$$

If $f, g\colon X \to Y$, then

$$f * g = (f *' f_0) * (f_0 *' g) = (f * f_0) *' (f_0 * g) = f *' g$$

and
$$g * f = (f_0 *' g) * (f *' f_0) = (f_0 * f) *' (g * f_0) = f *' g$$

Therefore $f * g = f *' g = g * f$. For associativity we have

$$(f * g) * h = (f * g) *' (f_0 * h) = (f *' f_0) * (g *' h) = f * (g * h) \quad \blacksquare$$

9 COROLLARY *If P is an H space and Q is any H cogroup, then $[Q;P]$ is an abelian group and the group structure is defined by the multiplication map in P.*

PROOF This follows on observing that the two laws of composition defined in $[Q;P]$ by using the comultiplication in Q or the multiplication in P satisfy the hypotheses of theorem 8. ■

Note that if P is just an H space (but not an H group), the law of composition in $[X;P]$ defined by the multiplication in P is in general not a group structure on $[X;P]$. However, if X is an H cogroup (for instance, a suspension), it follows from corollary 9 that this law of composition is a group structure on $[X;P]$, and in this case the resulting group structure on $[X;P]$ is the same no matter what multiplication map P is given (so long as it is an H space).

10 COROLLARY *If P is an H space, $\pi_n(P)$ is abelian for all $n \geq 1$ and the group structure in $\pi_n(P)$ is defined by the multiplication map in P.* ■

For a double suspension $S(SZ)$ whose points are represented in the form $[[z,t],t']$, with $z \in Z$ and $t, t' \in I$, there are two laws of composition in the set of maps $S(SZ) \to X$. If $f, g\colon S(SZ) \to X$, we define

$$(f * g)[[z,t], t'] = \begin{cases} f[[z,2t], t'] & 0 \leq t \leq \frac{1}{2} \\ g[[z, 2t-1], t'] & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and

$$(f *' g)[[z,t], t'] = \begin{cases} f[[z,t], 2t'] & 0 \leq t' \leq \frac{1}{2} \\ g[[z,t], 2t'-1] & \frac{1}{2} \leq t' \leq 1 \end{cases}$$

The corresponding operations in $[S(SZ);X]$ satisfy the hypotheses of theorem 8. Therefore they are equal, and $[S(SZ);X]$ is an abelian group. In particular, we have the following corollary.

11 COROLLARY *For $n \geq 2$, π_n is a functor to the category of abelian groups.* ■

A similar argument can be applied to the loop space ΩP, where P is itself an H space. There is a multiplication map in ΩP, because it is a loop space, and another multiplication obtained from the original multiplication in P. The corresponding laws of composition in $[X;\Omega P]$ satisfy theorem 8. Therefore it follows that if P is an H space, $\pi^{\Omega P}$ is a contravariant functor to the category of abelian groups.

7 THE FUNDAMENTAL GROUPOID

This section concerns paths in a topological space. This leads to another description (in Sec. 1.8) of the first homotopy group π_1, introduced in Sec. 1.6. We shall have occasion to define a number of homotopies between paths in a topological space. These homotopies are generalizations (to nonclosed paths) of those used in Sec. 1.5 to prove that a loop space is an H group and are defined by the same formulas (except that the t and t' arguments are interchanged). It is clear that this repetition of formulas could have been eliminated by proving a suitably general result about path spaces instead of merely considering loop spaces in Sec. 1.5. However, each usage has its own value, and it is hoped that the repetition may be an aid to understanding the formulas.

A *groupoid* is a small category in which every morphism is an equivalence. We list without proof a number of facts about groupoids which are easy consequences of general properties of categories.

1 *The relation between objects A and B of a groupoid defined by the condition* hom $(A,B) \neq \varnothing$ *is an equivalence relation.* ■

The equivalence classes of this equivalence relation are called the *components* of the groupoid. The groupoid is said to be *connected* if it has just one component.

2 *For any object A of a groupoid, the law of composition which sends $f, g\colon A \to A$ to $f \circ g\colon A \to A$ is a group operation in* hom (A,A). ■

3 *There is a covariant functor from any groupoid to the category of groups and homomorphisms which assigns to an object A the group* hom (A,A) *and to a morphism $f\colon A \to B$ the homomorphism*

$$h_f\colon \hom (A,A) \to \hom (B,B)$$

defined by $h_f(g) = f \circ g \circ f^{-1}$ for $g\colon A \to A$. ■

Because any morphism $f\colon A \to B$ in a groupoid is an equivalence, $h_f\colon \hom (A,A) \to \hom (B,B)$ is an isomorphism. The following statement describes the collection of isomorphisms obtained by taking all possible morphisms $f\colon A \to B$.

4 *If A and B are in the same component of a groupoid, the collection of isomorphisms $\{h_f \mid f\colon A \to B\}$ is a conjugacy class of isomorphisms* $\hom (A,A) \to \hom (B,B)$. ■

5 *Let F be a covariant functor from one groupoid $\mathcal{C}$ to another $\mathcal{C}'$. Then F maps each component of $\mathcal{C}$ into some component of $\mathcal{C}'$, and there is a natural transformation $F_*(A)$ from the covariant functor* $\hom_{\mathcal{C}} (A,A)$ *on $\mathcal{C}$ to the covariant functor* $\hom_{\mathcal{C}'} (F(A), F(A))$ *on $\mathcal{C}$ defined by*

$$F_*(A)(f) = F(f)\colon F(A) \to F(A) \qquad f\colon A \to A \quad ■$$

With these general remarks about groupoids out of the way, we proceed to define the fundamental groupoid. A *path* ω in a topological space is defined to be a continuous map $\omega: I \to X$ [note that the path is the map, not just the image set $\omega(I)$]. The *origin* of the path is the point $\omega(0)$, and the *end* of the path is the point $\omega(1)$. We also say that ω is a *path from* $\omega(0)$ *to* $\omega(1)$. A *closed path*, or *loop, at* $x_0 \in X$ is a path ω such that $\omega(0) = x_0 = \omega(1)$. If ω and ω' are paths in X such that end ω = orig ω', there is a *product path* $\omega * \omega'$ in X defined by the formula

$$(\omega * \omega')(t) = \begin{cases} \omega(2t) & 0 \le t \le 1/2 \\ \omega'(2t - 1) & 1/2 \le t \le 1 \end{cases}$$

Then orig $(\omega * \omega')$ = orig ω and end $(\omega * \omega')$ = end ω'.

We should like to form a category whose objects are the points of X, whose morphisms from x_1 to x_0 are the paths from x_0 to x_1, and with the composite defined to be the product path. With these definitions, neither axiom of a category is satisfied. That is, there need not be an identity morphism for each point, and it is generally not true that the associative law for product paths holds [that is, $\omega * (\omega' * \omega'')$ is usually different from $(\omega * \omega') * \omega''$]. A category can be obtained, however, if the morphisms are defined not to be the paths themselves, but instead, homotopy classes of paths.

Two paths ω and ω' in X are briefly said to be *homotopic*, denoted by $\omega \simeq \omega'$, if they are homotopic relative to $\dot{I}$. Thus a necessary condition that $\omega \simeq \omega'$ is that $\omega(0) = \omega'(0)$ and $\omega(1) = \omega'(1)$. For any $x_0, x_1 \in X$ the relation $\omega \simeq \omega'$ is an equivalence relation in the set of paths from x_0 to x_1. The resulting equivalence classes are called *path classes*, and if ω is a path in X, the path class containing it is denoted by $[\omega]$. Since two paths in the same path class have the same origin and the same end, we can speak of the origin and the end of a path class.

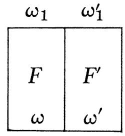
We shall construct a category whose objects are the points of X and whose morphisms from x_1 to x_0 are the path classes with x_0 as origin and x_1 as end. The following lemma shows that the path class of the product of two paths depends only on the path classes of the factors, and it will be used to define the composite in the category.

6 LEMMA *Let* $[\omega]$ *and* $[\omega']$ *be path classes in* X *with* end $[\omega]$ = orig $[\omega']$. *There is a well-defined path class* $[\omega] * [\omega'] = [\omega * \omega']$ *with* orig $([\omega] * [\omega'])$ = orig $[\omega]$ *and* end $([\omega] * [\omega'])$ = end $[\omega']$.

PROOF To prove that $\omega \simeq \omega_1$ and $\omega' \simeq \omega_1'$ imply $\omega * \omega' \simeq \omega_1 * \omega_1'$, let $F: I \times I \to X$ be a homotopy relative to $\dot{I}$ from ω to ω_1 and let $F': I \times I \to X$ be a homotopy relative to $\dot{I}$ from ω' to ω_1'. A homotopy $F * F': I \times I \to X$ is defined by the formula

$$(F * F')(t,t') = \begin{cases} F(2t,t') & 0 \le t \le 1/2 \\ F'(2t - 1, t') & 1/2 \le t \le 1 \end{cases}$$

and illustrated in the diagram



$F * F'$

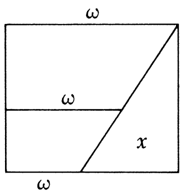
Then $F * F': \omega * \omega' \simeq \omega_1 * \omega_1' \text{ rel } \dot{I}$. ∎

7 THEOREM *For each topological space X there is a category $\mathcal{P}(X)$ whose objects are the points of X, whose morphisms from x_1 to x_0 are the path classes with x_0 as origin and x_1 as end, and whose composite is the product of path classes.*

PROOF To prove the existence of identity morphisms, let $\varepsilon_x: I \to X$ be the constant map of I to x for any $x \in X$. We show that $[\varepsilon_x] = 1_x$. If ω is a path with $\omega(1) = x$, we must prove that $\omega * \varepsilon_x \simeq \omega$ (with a similar property for paths with origin at x). Such a homotopy $F: I \times I \to X$ is defined by the formula

$$F(t,t') = \begin{cases} \omega\left(\dfrac{2t}{t'+1}\right) & 0 \leq t \leq \dfrac{t'+1}{2} \\ x & \dfrac{t'+1}{2} \leq t \leq 1 \end{cases}$$

and pictured in the diagram



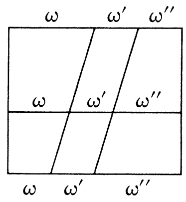
F

A similar homotopy shows that if $\omega(0) = x$, then $\varepsilon_x * \omega \simeq \omega$.

To prove the associativity of the composite of morphisms, let ω, ω', and ω'' be paths such that end ω = orig ω' and end ω' = orig ω''. We must prove that $(\omega * \omega') * \omega'' \simeq \omega * (\omega' * \omega'')$. Such a homotopy $G: I \times I \to X$ is defined by the formula

$$G(t,t') = \begin{cases} \omega\left(\dfrac{4t}{t'+1}\right) & 0 \leq t \leq \dfrac{t'+1}{4} \\ \omega'(4t - t' - 1) & \dfrac{t'+1}{4} \leq t \leq \dfrac{t'+2}{4} \\ \omega''\left(\dfrac{4t-2-t'}{2-t'}\right) & \dfrac{t'+2}{4} \leq t \leq 1 \end{cases}$$

and pictured in the diagram



G ∎

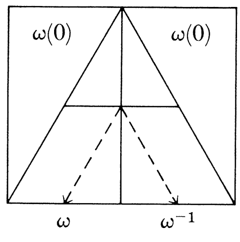
The category $\mathcal{P}(X)$ is called the *category of path classes* of X, or the *fundamental groupoid* of X, the latter because of the following theorem.

8 THEOREM *$\mathcal{P}(X)$ is a groupoid.*

PROOF Given a path ω in X, let $\omega^{-1}: I \to X$ be the path defined by $\omega^{-1}(t) = \omega(1-t)$. To prove that $[\omega^{-1}] = [\omega]^{-1}$ in $\mathcal{P}(X)$, we must show that $\omega * \omega^{-1} \simeq \varepsilon_{\omega(0)}$ [and also that $\omega^{-1} * \omega \simeq \varepsilon_{\omega(1)}$, which follows, however, from the first homotopy, because $\omega = (\omega^{-1})^{-1}$]. Such a homotopy $H: I \times I \to X$ is defined by the formula

$$H(t,t') = \begin{cases} \omega(0) & 0 \le t \le \dfrac{t'}{2} \\ \omega(2t - t') & \dfrac{t'}{2} \le t \le \dfrac{1}{2} \\ \omega(2 - 2t - t') & \dfrac{1}{2} \le t \le 1 - \dfrac{t'}{2} \\ \omega(0) & 1 - \dfrac{t'}{2} \le t \le 1 \end{cases}$$

and pictured in the diagram



H ∎

This completes the construction of the fundamental groupoid. The components of the fundamental groupoid are called *path components* of X. It is clear that x_0, and x_1 are in the same path component of X if and only if there is a path ω in X from x_0 to x_1. X is said to be *path connected* if its fundamental groupoid is connected. The following is an alternate characterization of the path components.

9 THEOREM *The path components of X are the maximal path-connected subspaces of X.*

PROOF Let A be a path component of X and let ω be a path in X such that $\omega(0) \in A$. We show that ω is a path in A. For each $t \in I$ define a path $\omega_t\colon I \to X$ by $\omega_t(t') = \omega(tt')$ for $t' \in I$. Then ω_t is a path in X from $\omega(0)$ to $\omega(t)$. Therefore $\omega(t)$ is in the same path component of X as x_0, namely A. Since this is so for every $t \in I$, ω is a path in A.

A is path connected because if $x_0, x_1 \in A$ there is a path ω in X from x_0 to x_1. By the above result, ω is a path in A. Therefore any two points of A can be joined by a path in A, and A is path connected. Since any path in X that starts in A stays in A, A is a maximal path-connected subset of X. ∎

10 LEMMA *A path-connected space is connected.*

PROOF If ω is a path in X, then $\omega(I)$, being a continuous image of the connected space I, is connected. Therefore $\omega(0)$ and $\omega(1)$ lie in the same component of X. If X is path connected, any two points of X lie in the same component, and X is connected. ∎

The converse of lemma 10 is false, as is shown by the following example.

11 EXAMPLE Let X be the subspace of $\mathbf{R}^2$ defined by

$$X = \{(x,y) \in \mathbf{R}^2 \mid x > 0,\ y = \sin\frac{1}{x} \quad \text{or} \quad x = 0,\ -1 \le y \le 1\}$$

Then X is connected, but not path connected.

Given a map $f\colon X \to Y$, there is a covariant functor $f_\#$ from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ which sends an object x of $\mathcal{P}(X)$ to the object $f(x)$ of $\mathcal{P}(Y)$ and the morphism $[\omega]$ of $\mathcal{P}(X)$ to the morphism $f_\#[\omega] = [f \circ \omega]$ of $\mathcal{P}(Y)$. The functorial properties of $f_\#$ are easily verified. From the first part of statement 5, or by direct verification, it follows that f maps each path component of X into some path component of Y. Therefore there is a covariant functor π_0 from the category of topological spaces and maps to the category of sets and functions such that $\pi_0(X)$ equals the set of path components of X, and

$$\pi_0(f) = f_\#\colon \pi_0(X) \to \pi_0(Y)$$

maps the path component of x in X to the path component of $f(x)$ in Y. If $F\colon f_0 \simeq f_1$, then for any $x \in X$ there is a path ω_x in Y from $f_0(x)$ to $f_1(x)$ defined by $\omega_x(t) = F(x,t)$ for $t \in I$. Therefore $f_0(x)$ and $f_1(x)$ belong to the same path component of Y, and $f_{0\#} = f_{1\#}$. It follows that π_0 can be regarded as a covariant functor from the homotopy category to the category of sets and functions. This functor characterizes the functor π_X for a contractible space X as follows.

12 THEOREM *If X is a contractible space, then π_X and π_0 are naturally equivalent functors on the homotopy category.*

PROOF If X and X' have the same homotopy type, then π_X and $\pi_{X'}$ are naturally equivalent. It follows from corollary 1.3.11 that if P is a one-point space, π_X is naturally equivalent to π_P. It therefore suffices to prove that π_P

is naturally equivalent to π_0. $\pi_0(P)$ consists of the single path component P, and a natural transformation

$$\psi\colon \pi_P \to \pi_0$$

is defined by $\psi[f] = f_\#(P)$ for $[f] \in [P;X]$. Because X^P is in one-to-one correspondence with X in such a way that homotopies $P \times I \to X$ correspond to paths $I \to X$, it follows that ψ is a natural equivalence. ∎

The functor π_0 is closely related to the functor H_0 of example 1.2.6. In fact, for spaces X whose components and path components coincide, H_0 is the composite of π_0 with the covariant functor which assigns to every set the free abelian group generated by that set. In particular, π_0 could have been used to obtain the results of Sec. 1.2 that were obtained by using H_0.

8 THE FUNDAMENTAL GROUP

By choosing a fixed point $x_0 \in X$ and considering the path classes in X with x_0 as origin and end, a group called the fundamental group is obtained. We show now that this group is naturally equivalent to the first homotopy group π_1, defined in Sec. 1.6. The section closes with a calculation of the fundamental group of the circle.

Let X be a topological space and let $x_0 \in X$. The *fundamental group of X based at x_0*, denoted by $\pi(X,x_0)$, is defined to be the group of path classes with x_0 as origin and end. It follows from theorem 1.7.8 and statement 1.7.2 that this is a group, and if $f\colon (X,x_0) \to (Y,y_0)$, then $f_\#$ is a homomorphism from $\pi(X,x_0)$ to $\pi(Y,y_0)$. If, f, $f'\colon (X,x_0) \to (Y,y_0)$ are homotopic, then

$$f_\# = f'_\#\colon \pi(X,x_0) \to \pi(Y,y_0).$$

Therefore, we have the following theorem.

1 THEOREM *There is a covariant functor from the homotopy category of pointed spaces to the category of groups which assigns to a pointed space its fundamental group and to a map f the homomorphism $f_\#$.* ∎

We show that the fundamental group functor π is naturally equivalent to π_1, defined in Sec. 1.6. Let $\lambda\colon I \to S(S^0)$ be defined by $\lambda(t) = [-1,t]$, where S^0 consists of the two points -1 and 1 and 1 is its basepoint. Then λ induces a bijection $\lambda^\#$ between the homotopy classes of maps $(S(S^0), 1) \to (X,x_0)$ and the path classes of closed paths in X at x_0 defined by

$$\lambda^\#[g] = [g\lambda] \qquad g\colon (S(S^0), 1) \to (X,x_0)$$

From the definition of the law of composition in $[S(S^0);X]$ and in $\pi(X,x_0)$, $\lambda^\#$ is seen to be a group isomorphism. Given a map $f\colon (X,x_0) \to (Y,y_0)$, $\lambda^\#$ commutes with $f_\#$. By lemma 1.6.6, $S(S^0)$ is homeomorphic to S^1.

2 THEOREM *The map $\chi^{\#}$ is a natural equivalence of the first homotopy group functor π_1 with the fundamental group functor π.* ∎

It will sometimes be convenient to regard the elements of $\pi(X,x_0)$ as homotopy classes of maps $(S^1,p_0) \to (X,x_0)$, rather than as path classes.

Because any closed path at x_0 (and any homotopy between such paths) must lie in the path component A of X containing x_0, it follows that $\pi(A,x_0) \approx \pi(X,x_0)$. Hence the fundamental group can give information only about the path component of X containing x_0. From general groupoid considerations (see statements 1.7.3 and 1.7.4), if $[\omega]$ is a path class in X from x_0 to x_1, then $h_{[\omega]}$ is an isomorphism from $\pi(X,x_1)$ to $\pi(X,x_0)$.

3 THEOREM *The fundamental groups of a path-connected space based at different points are isomorphic by an isomorphism determined up to conjugacy.* ∎

Even though the fundamental groups based at different points of a path-connected space are isomorphic, we cannot identify them, because the isomorphism between them is not unique. If the fundamental group at some point (and hence all points) is abelian, the isomorphism is unique. In general, the fundamental group need not be abelian; however, the following consequence of theorem 2 and corollary 1.6.10 is a general result about the commutativity of fundamental groups.

4 THEOREM *The fundamental group of a path-connected H space is abelian, and if ω and ω' are closed paths at the base point, then*

$$[\omega] * [\omega'] = [\mu \circ (\omega,\omega')]$$

where μ is the multiplication map in the H space. ∎

A space X is said to be *n-connected* for $n \geq 0$ if every continuous map $f\colon S^k \to X$ for $k \leq n$ has a continuous extension over E^{k+1}. A 1-connected space is also said to be *simply connected.* Note that if $0 \leq m \leq n$, an n-connected space is m-connected. It follows from theorem 1.6.7 that a space X is n-connected if and only if it is path connected and $\pi_k(X,x)$ is trivial for every base point $x \in X$ and $1 \leq k \leq n$. From corollary 1.3.13 we have the following result.

5 LEMMA *A contractible space is n-connected for every $n \geq 0$.* ∎

Note that a space is 0-connected if and only if it is path connected, and a space is simply connected if and only if it is path connected, and $\pi(X,x_0) = 0$ for some (and hence all) points $x_0 \in X$.

From theorem 1 we know that two pointed spaces having the same homotopy type as pointed spaces have isomorphic fundamental groups. To prove a similar result for two path-connected spaces which have the same homotopy type as spaces (no base-point condition) we need some preliminary results.

6 LEMMA *Let $h: I \times I \to X$ and let α_0, α_1, β_0, and β_1 be the paths in X defined by restricting h to the edges of $I \times I$ [that is, $\alpha_i(t) = h(i,t)$ and $\beta_i(t) = h(t,i)$]. Then $(\alpha_0 * \beta_1) * (\alpha_1{}^{-1} * \beta_0{}^{-1})$ is a closed path in X at $h(0,0)$ which represents the trivial element of $\pi(X, h(0,0))$.*

PROOF Let α_0', α_1', β_0', and β_1' be the paths in $I \times I$ defined by $\alpha_i'(t) = (i,t)$ and $\beta_i'(t) = (t,i)$. Then $(\alpha_0' * \beta_1') * (\alpha_1'^{-1} * \beta_0'^{-1})$ is a closed path in $I \times I$ at $(0,0)$ and h maps this closed path into $(\alpha_0 * \beta_1) * (\alpha_1{}^{-1} * \beta_0{}^{-1})$. Since $I \times I$ is a convex subset of $\mathbf{R}^2$, it is contractible, and by lemma 5, it is simply connected. Therefore

$$(\alpha_0' * \beta_1') * (\alpha_1'^{-1} * \beta_0'^{-1}) \simeq \varepsilon_{(0,0)}$$

and

$$\begin{aligned}(\alpha_0 * \beta_1) * (\alpha^{-1} * \beta_0{}^{-1}) &= h \circ ((\alpha_0' * \beta_1') * (\alpha_1'^{-1} * \beta_0'^{-1})) \\ &\simeq h \circ \varepsilon_{(0,0)} = \varepsilon_{h(0,0)} \quad \blacksquare\end{aligned}$$

7 THEOREM *Let $f: (X,x_0) \to (Y,y_0)$ and $g: (X,x_0) \to (Y,y_1)$ be homotopic as maps of X to Y. Then there is a path ω in Y from y_0 to y_1 such that*

$$f_\# = h_{[\omega]} \circ g_\#: \pi(X,x_0) \to \pi(Y,y_0)$$

PROOF Let $F: X \times I \to Y$ be a homotopy from f to g and let $\omega: I \to Y$ be defined by $\omega(t) = F(x_0,t)$. Then ω is a path in Y from y_0 to y_1. If ω' is any closed path in X at x_0, let $h: I \times I \to Y$ be defined by $h(t,t') = F(\omega'(t), t')$. Then $h(0,t') = F(x_0,t') = \omega(t')$, $h(t,1) = g\omega'(t)$, $h(1,t') = \omega(t')$, and $h(t,0) = f\omega'(t)$. By lemma 6 we have

$$(\omega * g\omega') * (\omega^{-1} * (f\omega')^{-1}) \simeq \varepsilon_{y_0}$$

This implies $[\omega] \circ g_\#[\omega'] \circ [\omega]^{-1} = f_\#[\omega']$, or $(h_{[\omega]} \circ g_\#)[\omega'] = f_\#[\omega']$. Since $[\omega']$ is an arbitrary element of $\pi(X,x_0)$, $h_{[\omega]} \circ g_\# = f_\#$. $\blacksquare$

8 THEOREM *Two path-connected spaces with the same homotopy type have isomorphic fundamental groups.*

PROOF Let $f: X \to Y$ be a homotopy equivalence with homotopy inverse $g: Y \to X$. Let $x_0 \in X$ and set $y_0 = f(x_0)$, $x_1 = g(y_0)$, and $y_1 = f(x_1)$. Let $f_0: (X,x_0) \to (Y,y_0)$ and $f_1: (X,x_1) \to (Y,y_1)$ be maps defined by f (that is, f_0 and f_1 are both equal to f but are regarded as maps of pairs), and let $g': (Y,y_0) \to (X,x_1)$ be defined by g. Then $g' \circ f_0: (X,x_0) \to (X,x_1)$ is homotopic, as a map of X to X, to $1_{(X,x_0)}: (X,x_0) \subset (X,x_0)$, and $f_1 \circ g': (Y,y_0) \to (Y,y_1)$ is homotopic, as a map of Y to Y, to $1_{(Y,y_0)}: (Y,y_0) \subset (Y,y_0)$. It follows from theorem 7 that there are paths ω in X from x_1 to x_0 and ω' in Y from y_1 to y_0 such that

$$h_{[\omega]} = (g' \circ f_0)_\# = g'_\# \circ f_{0\#} \qquad \text{and} \qquad h_{[\omega']} = (f_1 \circ g')_\# = f_{1\#} \circ g'_\#$$

Therefore we have a commutative diagram

$$\begin{tikzcd} \pi(X,x_0) \arrow[r, "h_{[\omega]}"] \arrow[d, "f_{0\#}"'] & \pi(X,x_1) \arrow[d, "f_{1\#}"] \\ \pi(Y,y_0) \arrow[r, "h_{[\omega']}"'] \arrow[ur, "g'_\#"] & \pi(Y,y_1) \end{tikzcd}$$

$g'_\#$ is an epimorphism because $h_{[\omega]}$ is, and it is a monomorphism because $h_{[\omega']}$ is. Therefore $g'_\#$ is an isomorphism. ∎

We close with an example of a space with a nontrivial fundamental group. For this purpose we compute $\pi(S^1,p_0)$ following a method used by Tucker[1], where $S^1 = \{e^{i\theta}\}$ and $p_0 = 1$.

The *exponential map* ex: $\mathbf{R} \to S^1$ is defined by $ex(t) = e^{2\pi it}$. Then ex is continuous, $ex(t_1 + t_2) = ex(t_1)\, ex(t_2)$ (where the right-hand side is multiplication of complex numbers), and $ex(t_1) = ex(t_2)$ if and only if $t_1 - t_2$ is an integer. It follows that $ex|(-\frac{1}{2},\frac{1}{2})$ is a homeomorphism of the open interval $(-\frac{1}{2},\frac{1}{2})$ onto $S^1 - \{e^{\pi i}\}$. We let

$$lg\colon S^1 - \{e^{\pi i}\} \to (-\tfrac{1}{2},\tfrac{1}{2})$$

be the inverse of $ex|(-\frac{1}{2},\frac{1}{2})$.

A subset $X \subset \mathbf{R}^n$ will be called *starlike* from a point $x_0 \in X$ if, whenever $x \in X$, the closed line segment $[x_0,x]$ from x_0 to x lies in X.

9 LEMMA *Let X be compact and starlike from $x_0 \in X$. Given any continuous map f: $X \to S^1$ and any $t_0 \in \mathbf{R}$ such that $ex(t_0) = f(x_0)$, there exists a continuous map f': $X \to \mathbf{R}$ such that $f'(x_0) = t_0$ and $ex(f'(x)) = f(x)$ for all $x \in X$.*

PROOF Clearly, we can translate X so that it is starlike from the origin; hence there is no loss of generality in assuming $x_0 = 0$. Since X is compact, f is uniformly continuous and there exists $\varepsilon > 0$ such that if $\|x - x'\| < \varepsilon$, then $\|f(x) - f(x')\| < 2$ [that is, $f(x)$ and $f(x')$ are not antipodes in S^1]. Since X is bounded, there exists a positive integer n such that $\|x\|/n < \varepsilon$ for all $x \in X$. Then for each $0 \le j < n$ and all $x \in X$

$$\left\|\frac{(j+1)x}{n} - \frac{jx}{n}\right\| = \frac{\|x\|}{n} < \varepsilon$$

and so

$$\left\|f\left(\frac{(j+1)x}{n}\right) - f\left(\frac{jx}{n}\right)\right\| < 2$$

It follows that the quotient $f((j+1)x/n)/f(jx/n)$ is a point of $S^1 - \{e^{\pi i}\}$. Let g_j: $X \to S^1 - \{e^{\pi i}\}$ for $0 \le j < n$ be the map defined by $g_j(x) =$

[1] See A. W. Tucker, Some topological properties of disk and sphere, *Proceedings of the Canadian Mathematical Congress*, 1945, pp. 285–309.

$f((j+1)x/n)/f(jx/n)$. Then, for all $x \in X$, we see that

$$f(x) = f(0)g_0(x)g_1(x) \cdots g_{n-1}(x)$$

We define f': $X \to \mathbf{R}$ by

$$f'(x) = t_0 + lg(g_0(x)) + lg(g_1(x)) + \cdots + lg(g_{n-1}(x))$$

f' is the sum of $n+1$ continuous functions from X to $\mathbf{R}$, so it is continuous. Clearly, $f'(0) = t_0$ and $ex(f'(x)) = f(x)$. ■

10 LEMMA *Let X be a connected space and let f', g': $X \to \mathbf{R}$ be maps such that $ex \circ f' = ex \circ g'$ and $f'(x_0) = g'(x_0)$ for some $x_0 \in X$. Then $f' = g'$.*

PROOF Let $h = f' - g'$: $X \to \mathbf{R}$. Since $ex \circ f' = ex \circ g'$, $ex \circ h$ is the constant map of X to p_0. Therefore h is a continuous map of X to $\mathbf{R}$, taking only integral values. Because X is connected, h is constant, and since $h(x_0) = 0$, $h(x) = 0$ for all $x \in X$. ■

Let α: $I \to S^1$ be a closed path at p_0. Because I is starlike from 0 and $\alpha(0) = p_0 = ex(0)$, it follows from lemma 9 that there exists α': $I \to \mathbf{R}$ such that $\alpha'(0) = 0$ and $ex \circ \alpha' = \alpha$. By lemma 10, α' is uniquely characterized by these properties. Because $ex(\alpha'(1)) = p_0$, it follows that $\alpha'(1)$ is an integer. We define the *degree* of α by $\deg \alpha = \alpha'(1)$.

11 LEMMA *Let α and β be homotopic closed paths in S^1 at p_0. Then* $\deg \alpha = \deg \beta$.

PROOF Let F: $I \times I \to S^1$ be a homotopy relative to $\dot{I}$ from α to β. Because $I \times I$ is a starlike subset of $\mathbf{R}^2$ from (0,0), it follows from lemma 9 that there is a map F': $I \times I \to \mathbf{R}$ such that $F'(0,0) = 0$ and $ex \circ F' = F$. Since F is a homotopy relative to $\dot{I}$, $F(0,t') = F(1,t') = p_0$ for all $t \in I$. Therefore $F'(0,t')$ and $F'(1,t')$ take on only integral values for all $t' \in I$. It follows that $F'(0,t')$ must be constant and $F'(1,t')$ must be constant. Because $F'(0,0) = 0$, $F'(0,t') = 0$ for all $t' \in I$. Define α', β': $I \to \mathbf{R}$ by $\alpha'(t) = F'(t,0)$ and $\beta'(t) = F'(t,1)$. Then $\alpha'(0) = 0$ and $ex \circ \alpha' = \alpha$. Therefore $\deg \alpha = \alpha'(1) = F'(1,0)$. Similarly, $\beta'(0) = 0$ and $ex \circ \beta' = \beta$, so $\deg \beta = \beta'(1) = F'(1,1)$. Because $F'(1,t')$ is constant, $F'(1,0) = F'(1,1)$ and $\deg \alpha = \deg \beta$. ■

It follows that there is a well-defined function *deg* from $\pi(S^1,p_0)$ to $\mathbf{Z}$ defined by

$$deg\,[\alpha] = \deg \alpha$$

where α is a closed path in S^1 at p_0.

12 THEOREM *The function deg is an isomorphism*

$$deg\colon \pi(S^1,p_0) \approx \mathbf{Z}$$

PROOF To prove that *deg* is a homomorphism, let α and β be two closed paths in S^1 at p_0 and let $\alpha\beta$ be the closed path which is their pointwise product in the group multiplication of S^1. We know from theorem 4 that $[\alpha] * [\beta] = [\alpha\beta]$. Let α', β': $I \to \mathbf{R}$ be such that $\alpha'(0) = 0$, $ex \circ \alpha' = \alpha$, $\beta'(0) = 0$, and $ex \circ \beta' = \beta$. Then $\alpha' + \beta'$: $I \to \mathbf{R}$ is such that $(\alpha' + \beta')(0) = 0$ and $ex \circ (\alpha' + \beta') = \alpha\beta$. Therefore

$$\begin{aligned} deg\,([\alpha] * [\beta]) &= deg\,[\alpha\beta] = (\alpha' + \beta')(1) \\ &= \deg \alpha + \deg \beta = deg\,[\alpha] + deg\,[\beta] \end{aligned}$$

showing that deg is a homomorphism.

The map deg is an epimorphism; for if n is an integer, there is a path α_n' in $\mathbf{R}$ defined by $\alpha_n'(t) = tn$. Let $\alpha_n = ex \circ \alpha_n'$. Then clearly, $deg\,[\alpha_n] = \alpha_n'(1) = n$.

The map deg is a monomorphism; for if $deg\,[\alpha] = 0$, there is a closed path α' in $\mathbf{R}$ at 0 such that $ex \circ \alpha' = \alpha$. Since $\mathbf{R}$ is simply connected (because it is contractible, and by lemma 5), $\alpha' \simeq \varepsilon_0$. Then $ex \circ \alpha' \simeq \varepsilon_{p_0}$. Therefore $\alpha \simeq \varepsilon_{p_0}$ and $[\alpha]$ is the identity element of $\pi(S^1,p_0)$. ■

The method we have used to compute $\pi(S^1,p_0)$ will be generalized in Chapter 2 to give relations between the fundamental group of a space and the fundamental groups of its covering spaces.

It follows from theorem 2 that $\pi(S^1,p_0) \approx [S^1,p_0;\, S^1,p_0]$. Because S^1 is a topological group, the set $[S^1;S^1]$ (with no base-point condition) is also a group under pointwise multiplication of maps, and there is an obvious homomorphism

$$\gamma\colon [S^1,p_0;\, S^1,p_0] \to [S^1;S^1]$$

13 LEMMA *The homomorphism*

$$\gamma\colon [S^1,p_0;\, S^1,p_0] \to [S^1;S^1]$$

is an isomorphism.

PROOF To show that γ is an epimorphism, let f: $S^1 \to S^1$ and let $f(p_0) = e^{i\theta}$ for some $0 \le \theta < 2\pi$. Define a homotopy F: $S^1 \times I \to S^1$ by

$$F(z,t) = f(z)e^{-it\theta}$$

Then F is a homotopy from f to a map f' such that $f'(p_0) = p_0$. Therefore $\gamma[f']_{p_0} = [f'] = [f]$.

To show that γ is a monomorphism, assume that f: $(S^1,p_0) \to (S^1,p_0)$ is such that $\gamma[f]_{p_0} = [f]$ is trivial. Then f: $S^1 \to S^1$ is null homotopic. By theorem 1.3.12, f is null homotopic relative to p_0. Therefore $[f]_{p_0}$ is trivial. ■

It follows from theorem 12 and lemma 13 that $[S^1;S^1] \approx \mathbf{Z}$. The isomorphism can be chosen so that for each integer n the map $z \to z^n$ from S^1 to itself represents a homotopy class corresponding to n.

EXERCISES

A CONTRACTIBLE SPACES

1 The *cone* over a topological space X with *vertex* v is defined to be the mapping cylinder of the constant map $X \to v$. Prove that X is contractible if and only if it is a retract of any cone over X.

2 Prove that S^n is a retract of E^{n+1} if and only if S^n is contractible.

3 If CX is a cone over X, prove that (CX,X) has the homotopy extension property with respect to any space.

4 Prove that a space Y is contractible if and only if, given a pair (X,A) having the homotopy extension property with respect to Y, any map $A \to Y$ can be extended over X.

5 Let Y be the comb space of example 1.3.9 and let y_0 be the point $(0,1) \in Y$. Let Y' be another copy of Y, with corresponding point y_0'. Let X be the space obtained by forming the disjoint union of Y and Y' and identifying y_0 with y_0'. Prove that X is n-connected for all n but not contractible. (*Hint:* Any deformation of X in itself must be a homotopy relative to y_0.)

B ADJUNCTION SPACES

1 Let A be a subspace of a space X and let $f\colon A \to Y$ be a continuous map. The *adjunction space* Z of X to Y by f is defined to be the quotient space of the disjoint union of X and Y by the identifications $x \in A$ equals $f(x) \in Y$ for all $x \in A$. Prove that if X and Y are normal spaces and A is closed in X, then Z is a normal space.

2 A space X is said to be *binormal* if $X \times I$ is a normal space. If X is a binormal space, Y is a normal space, and $f\colon X \to Y$ is continuous, prove that the mapping cylinder of f is a normal space.

3 Given a continuous map $f\colon A \to Y$, where A is a subspace of a space X, prove that f can be extended over X if and only if Y is a retract of the adjunction space of X to Y by f.

4 Let Z be the adjunction space of X to Y by a map $f\colon A \to Y$. Prove that (Z,Y) has the homotopy extension property with respect to a space W if (X,A) has the homotopy extension property with respect to W.

C ABSOLUTE RETRACTS AND ABSOLUTE NEIGHBORHOOD RETRACTS

A space Y is said to be an *absolute retract* (or *absolute neighborhood retract*) if, given a normal space X, closed subset $A \subset X$, and a continuous map $f\colon A \to Y$, then f can be extended over X (or f can be extended over some neighborhood of A in X).

1 Prove that a normal space Y is an absolute retract (or absolute neighborhood retract) if and only if, whenever Y is imbedded as a closed subset of a normal space Z, then Y is a retract of Z (or a retract of some neighborhood of Y in Z).

2 Prove that the product of arbitrarily many absolute retracts (or finitely many absolute neighborhood retracts) is itself an absolute retract (or absolute neighborhood retract).

3 Prove that $\mathbf{R}^n$ is an absolute retract for all n.

4 Prove that a retract of an absolute retract is an absolute retract and that a retract of some open subset of an absolute neighborhood retract is an absolute neighborhood retract.

5 Prove that E^n is an absolute retract and S^n is an absolute neighborhood retract for all n.

6 Prove that a binormal absolute neighborhood retract is *locally contractible* (that is, every neighborhood U of a point x contains a neighborhood V of x deformable to x in U).

7 Prove that a binormal absolute neighborhood retract is an absolute retract if and only if it is contractible.

D HOMOTOPY EXTENSION PROPERTY

1 Let A be a closed subset of a normal space X, let $f\colon X \to Y$ be continuous (where Y is arbitrary), and let $G\colon A \times I \to Y$ be a homotopy of $f \mid A$. If there exists a homotopy $G'\colon U \times I \to Y$ of $f \mid U$ which extends G, where U is an open neighborhood of A, show that there exists a homotopy $F\colon X \times I \to Y$ of f which extends G.

2 *Borsuk's homotopy extension theorem.* Let A be a closed subspace of a binormal space X. Then (X,A) has the homotopy extension property with respect to any absolute neighborhood retract Y.

3 Let A be a closed subset of a binormal space X and assume that the subspace $A \times I \cup X \times 0 \subset X \times I$ is an absolute neighborhood retract. Then (X,A) has the homotopy extension property with respect to any space Y.

4 Let A be a closed subset of X and B a subset of Y. Assume that (X,A) has the homotopy extension property with respect to B and that $(X \times I, X \times \dot{I} \cup A \times I)$ has the homotopy extension property with respect to Y. Prove that if $f\colon (X,A) \to (Y,B)$ is homotopic (as a map of pairs) to a map which sends all of X to B, then it is homotopic relative to A to such a map.

E COFIBRATIONS

1 Prove that any cofibration is an injective function.

2 Prove that a composite of cofibrations is a cofibration.

3 For a closed subspace A of X prove that the inclusion map $A \subset X$ is a cofibration if and only if $X \times 0 \cup A \times I$ is a retract of $X \times I$.

4 If A is a subspace of a Hausdorff space X, prove that if $A \subset X$ is a cofibration, then A is closed in X.

5 Assume that X is the union of closed subsets X_1 and X_2 and let A be a subset of X such that $X_1 \cap X_2 \subset A$. Prove that if $A \cap X_1 \subset X_1$ and $A \cap X_2 \subset X_2$ are cofibrations, so is $A \subset X$.

6 Let A be a closed subspace of a space X. Prove that the following are equivalent:[1]

(*a*) $A \subset X$ is a cofibration.
(*b*) There is a deformation $D\colon X \times I \to X$ rel A [that is, $D(x,0) = x$ and $D(a,t) = a$ for $x \in X$, $a \in A$, and $t \in I$] and a map $\varphi\colon X \to I$ such that $A = \varphi^{-1}(1)$ and $D(\varphi^{-1}(0,1] \times 1) \subset A$.

[1] If X is normal, the equivalence of (*a*) and (*c*) is proved in G. S. Young, A condition for the absolute homotopy extension property, *American Mathematical Monthly*, vol. 71, pp. 896–897, 1964.

(*c*) There is a neighborhood U of A deformable in X to A rel A [that is, there is a homotopy $H: U \times I \to X$ such that $H(x,0) = x$, $H(a,t) = a$, and $H(x,1) \in A$ for $x \in U$, $a \in A$, and $t \in I$] and a map $\varphi: X \to I$ such that $A = \varphi^{-1}(1)$ and $\varphi(x) = 0$ if $x \in X - U$.

7 If $A \subset X$ and $B \subset Y$ are cofibrations with A and B closed in X and Y, respectively, prove that $A \times B \subset X \times B \cup A \times Y$ and $X \times B \cup A \times Y \subset X \times Y$ are cofibrations.

F LOCAL SYSTEMS[1]

1 A *local system* on a space X is a covariant functor from the fundamental groupoid of X to some category. For any category $\mathcal{C}$ show that there is a category of local systems on X with values in $\mathcal{C}$. (Two local systems on X are said to be *equivalent* if they are equivalent objects in this category.)

2 Given a map $f: X \to Y$, show that f induces a covariant functor from the category of local systems on Y with values in $\mathcal{C}$ to the category of local systems on X with values in $\mathcal{C}$.

3 If A is an object of a category $\mathcal{C}$, let Aut A be the group of self-equivalences of A in $\mathcal{C}$. If $\varphi: A \approx B$ is an equivalence in $\mathcal{C}$, then show that $\bar{\varphi}$: Aut $A \to$ Aut B defined by $\bar{\varphi}(\alpha) = \varphi \circ \alpha \circ \varphi^{-1}$ is an isomorphism of groups.

4 If Γ is a local system on X and $x_0 \in X$, show that Γ induces a homomorphism

$$\bar{\Gamma}_{x_0}: \pi(X,x_0) \to \operatorname{Aut} \Gamma(x_0)$$

5 If X is path connected, prove that two local systems Γ and Γ' on X with values in $\mathcal{C}$ are equivalent if and only if there is an equivalence $\varphi: \Gamma(x_0) \approx \Gamma'(x_0)$, such that $\bar{\varphi} \circ \bar{\Gamma}_{x_0}$ is conjugate to $\bar{\Gamma}'_{x_0}$ in Aut $\Gamma'(x_0)$.

6 If X is path connected, given an object $A \in \mathcal{C}$ and a homomorphism $\alpha: \pi(X,x_0) \to$ Aut A, prove that there is a local system Γ on X with values in $\mathcal{C}$ such that $\Gamma(x_0) = A$ and $\bar{\Gamma}_{x_0} = \alpha$.

G THE FUNDAMENTAL GROUP

1 Prove that the fundamental group functor commutes with direct products.

2 If ω and ω' are paths in X from x_0 to x_1, prove that $\omega \simeq \omega'$ if and only if $\omega * \omega'^{-1} \simeq \varepsilon_{x_0}$.

3 Let a space X be the union of two open simply connected subsets U and V such that $U \cap V$ is nonempty and path connected. Prove that X is simply connected.

4 Prove that S^n is simply connected for $n \geq 2$.

5 If there exists a space with a nonabelian fundamental group, prove that the "figure eight" (that is, the union of two circles with a point in common) has a nonabelian fundamental group (cf. exercise 2.B.4).

6 Let $f: I \to \mathbf{R}^2$ be a continuously differentiable simple closed curve in the plane with a nowhere-vanishing tangent vector [that is, $f(0) = f(1)$, $f'(0) = f'(1)$, and $f'(t) \neq 0$ for

[1] See N. E. Steenrod, Homology with local coefficients, *Annals of Mathematics*, vol. 44, pp. 610–627, 1943.

$0 \leq t \leq 1]$. Let $\omega: I \to S^1$ be the closed path defined by $\omega(t) = f'(t)/\|f'(t)\|$. Prove that $[\omega]$ is a generator of $\pi(S^1)$.[1]

7 In $\mathbf{R}^2$, let X be the space consisting of the union of the circles C_n, where C_n has center $(1/n,0)$ and radius $1/n$ for all positive integers n. In $\mathbf{R}^3$ (with $\mathbf{R}^2$ imbedded as the plane $x_3 = 0$), let Y be the set of points on the closed line segments joining $(0,0,1)$ to X and let Y' be the reflection of Y through the origin of $\mathbf{R}^3$. Then Y and Y' are closed simply connected subsets of $\mathbf{R}^3$ such that $Y \cap Y'$ is a single point. Prove that $Y \cup Y'$ is not simply connected.[2]

H SOME APPLICATIONS OF THE FUNDAMENTAL GROUP

1 Prove that S^1 is not a retract of E^2.

2 Prove that S^1 and S^n for $n \geq 2$ are not of the same homotopy type.

3 Prove that $\mathbf{R}^2$ and $\mathbf{R}^n$ for $n > 2$ are not homeomorphic.

4 Let $p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n, having complex coefficients and leading coefficient 1, and let $q(z) = z^n$. For $r > 0$ let $C_r = \{x \in \mathbf{R}^2 \mid \|x\| = r\}$. Prove that for r large enough, $p \mid C_r$ and $q \mid C_r$ are homotopic maps of C_r into $\mathbf{R}^2 - 0$.

5 *Fundamental theorem of algebra.* Prove that every complex polynomial has a root. (*Hint:* For any $r > 0$ the map $q \mid C_r$: $C_r \to \mathbf{R}^2 - 0$ is not null homotopic because it induces a nontrivial homomorphism of fundamental groups.)

[1] See H. Hopf, Uber die Drehung der Tangenten und Sehnen ebener Kurven, *Compositio Mathematica*, vol. 2, pp. 50–62, 1935. For generalizations see H. Whitney, On regular closed curves in the plane, *Compositio Mathematica*, vol. 4, pp. 276–284, 1937, and S. Smale, Regular curves on Riemannian manifolds, *Transactions of the American Mathematical Society*, vol. 87, pp. 495–512, 1958.

[2] See H. B. Griffiths, The Fundamental group of two spaces with a common point, *Quarterly Journal of Mathematics*, vol. 5, pp. 175–190, 1954.

CHAPTER TWO
COVERING SPACES AND FIBRATIONS

THE THEORY OF COVERING SPACES IS IMPORTANT NOT ONLY IN TOPOLOGY, BUT also in differential geometry, complex analysis, and Lie groups. The theory is presented here because the fundamental group functor provides a faithful representation of covering-space problems in terms of algebraic ones. This justifies our special interest in the fundamental group functor.

This chapter contains the theory of covering spaces, as well as an introduction to the related concepts of fiber bundle and fibration. These concepts will be considered again later in other contexts. Here we adopt the view that certain fibrations, namely, those having the property of unique path lifting, are generalized covering spaces. Because of this, we shall consider these fibrations in some detail.

Covering spaces are defined in Sec. 2.1, and fibrations are defined in Sec. 2.2, where it is proved that every covering space is a fibration. Section 2.3 deals with relations between the fundamental groups of the total space and base space of a fibration with unique path lifting, and Sec. 2.4 contains a solution of the lifting problem for such fibrations in terms of the fundamental group functor.

The lifting theorem is applied in Sec. 2.5 to classify the covering spaces of a connected locally path-connected space by means of subgroups of its fundamental group. This entails the construction of a covering space starting with the base space and a subgroup of its fundamental group. In Sec. 2.6 a converse problem is considered. The base space is constructed, starting with a covering space and a suitable group of transformations on it.

In Sec. 2.7 fiber bundles are introduced as natural generalizations of covering spaces. The main result of the section is that local fibrations are fibrations. This implies that a fiber bundle with paracompact base space is a fibration. Section 2.8 considers properties of general fibrations and the concept of fiber homotopy equivalence. These will be important in our later study of homotopy theory.

1 COVERING PROJECTIONS

A covering projection is a continuous map that is a uniform local homeomorphism. This and related concepts are introduced in this section, along with some examples and elementary properties.

Let $p: \tilde{X} \to X$ be a continuous map. An open subset $U \subset X$ is said to be *evenly covered* by p if $p^{-1}(U)$ is the disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. If U is evenly covered by p, it is clear that any open subset of U is also evenly covered by p. A continuous map $p: \tilde{X} \to X$ is called a *covering projection* if each point $x \in X$ has an open neighborhood evenly covered by p. $\tilde{X}$ is called the *covering space* and X the *base space* of the covering projection.

The following are examples of covering projections.

1 Any homeomorphism is a covering projection.

2 If $\tilde{X}$ is the product of X with a discrete space, the projection $\tilde{X} \to X$ is a covering projection.

3 The map $ex: \mathbf{R} \to S^1$, defined by $ex(t) = e^{2\pi i t}$, (considered in Sec 1.8) is a covering projection.

4 For any positive integer n the map $p: S^1 \to S^1$, defined by $p(z) = z^n$, is a covering projection.

5 For any integer $n \geq 1$ the map $p: S^n \to P^n$, which identifies antipodal points, is a covering projection.

6 If G is a topological group, H is a discrete subgroup of G, and G/H is the space of left (or right) cosets, then the projection $G \to G/H$ is a covering projection.

A continuous map $f: Y \to X$ is called a *local homeomorphism* if each point $y \in Y$ has an open neighborhood mapped homeomorphically by f onto an open subset of X. If this is so, each point of Y has arbitrarily small neighborhoods with this property, and we have the following lemmas.

7 LEMMA *A local homeomorphism is an open map.* ∎

8 LEMMA *A covering projection is a local homeomorphism.*

PROOF Let $p: \tilde{X} \to X$ be a covering projection and let $\tilde{x} \in \tilde{X}$. Let U be an open neighborhood of $p(\tilde{x})$ evenly covered by p. Then $p^{-1}(U)$ is the disjoint union of open sets, each mapped homeomorphically onto U by p. Let $\tilde{U}$ be that one of these open sets which contains $\tilde{x}$. Then $\tilde{U}$ is an open neighborhood of $\tilde{x}$ such that $p \mid \tilde{U}$ is a homeomorphism of $\tilde{U}$ onto the open subset U of X. ∎

A local homeomorphism need not be a covering projection, as shown by the following example.

9 EXAMPLE Let $p: (0,3) \to S^1$ be the restriction of the map $ex: \mathbf{R} \to S^1$ of example 3 to the open interval $(0,3)$. Because p is the restriction of a local homeomorphism to an open subset, it is a local homeomorphism. It is also a surjection, but it is not a covering projection because the complex number $1 \in S^1$ has no neighborhood evenly covered by p.

The following is a consequence of lemmas 7 and 8 and the fact (immediate from the definition) that a covering projection is a surjection.

10 COROLLARY *A covering projection exhibits its base space as a quotient space of its covering space.* ∎

For locally connected spaces there is the following reduction of covering projections to the components of the base space.

11 THEOREM *If X is locally connected, a continuous map $p: \tilde{X} \to X$ is a covering projection if and only if for each component C of X the map*

$$p \mid p^{-1}C: p^{-1}C \to C$$

is a covering projection.

PROOF If p is a covering projection and C is a component of X, let $x \in C$ and let U be an open neighborhood of x evenly covered by p. Let V be the component of U containing x. Since X is locally connected, V is open in X, and hence open in C. Clearly, V is evenly covered by $p \mid p^{-1}C$. Therefore $p \mid p^{-1}C$ is a covering projection.

Conversely, assume that the map $p \mid p^{-1}C: p^{-1}C \to C$ is a covering projection for each component C of X. Let $x \in C$ and let U be an open neighborhood of x in C evenly covered by $p \mid p^{-1}C$. Since X is locally connected, C is open in X. Therefore U is also open in X and is clearly evenly covered by p. Hence p is a covering projection. ∎

In general, the representation of the inverse image of an evenly covered open set as a disjoint union of open sets, each mapped homeomorphically, is not unique (consider the case of an evenly covered discrete set); however, for connected evenly covered open sets there is the following characterization of these open subsets.

12 LEMMA *Let U be an open connected subset of X which is evenly covered by a continuous map $p\colon \tilde{X} \to X$. Then p maps each component of $p^{-1}(U)$ homeomorphically onto U.*

PROOF By assumption, $p^{-1}(U)$ is the disjoint union of open subsets, each mapped homeomorphically onto U by p. Since U is connected, each of these open subsets is also connected. Because they are open and disjoint, each is a component of $p^{-1}(U)$. ∎

13 COROLLARY *Consider a commutative triangle*

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "p"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

where X is locally connected and p_1 and p_2 are covering projections. If p is a surjection, it is a covering projection.

PROOF If U is a connected open subset of X which is evenly covered by p_1 and p_2, it follows easily from lemma 12 that each component of $p_2{}^{-1}(U)$ is evenly covered by p. ∎

14 THEOREM *If $p\colon \tilde{X} \to X$ is a covering projection onto a locally connected base space, then for any component $\tilde{C}$ of $\tilde{X}$ the map*

$$p \mid \tilde{C}\colon \tilde{C} \to p(\tilde{C})$$

is a covering projection onto some component of X.

PROOF *Let* $\tilde{C}$ be a component of $\tilde{X}$. We show that $p(\tilde{C})$ is a component of X. $p(\tilde{C})$ is connected; to show that it is an open and closed subset of X, let x be in the closure of $p(\tilde{C})$ and let U be an open connected neighborhood of x evenly covered by p. Because U meets $p(\tilde{C})$, $p^{-1}(U)$ meets $\tilde{C}$. Therefore some component $\tilde{U}$ of $p^{-1}(U)$ meets $\tilde{C}$. Since $\tilde{C}$ is a component of $\tilde{X}$, $\tilde{U} \subset \tilde{C}$. Then, by lemma 12, $p(\tilde{C}) \supset p(\tilde{U}) = U$. Therefore the closure of $p(\tilde{C})$ is contained in the interior of $p(\tilde{C})$, which implies that $p(\tilde{C})$ is open and closed. The same argument shows that if $x \in p(\tilde{C})$ and U is an open connected neighborhood of x in X evenly covered by p, then $U \subset p(\tilde{C})$ and $(p \mid \tilde{C})^{-1}(U)$ is the disjoint union of those components of $p^{-1}(U)$ that meet $\tilde{C}$. It follows from lemma 12 that U is evenly covered by $p \mid \tilde{C}$. Therefore $p \mid \tilde{C}\colon \tilde{C} \to p(\tilde{C})$ is a covering projection. ∎

The following example shows that the converse of theorem 14 is false.

15 EXAMPLE Let $X = S^1 \times S^1 \times \cdots$ be a countable product of 1-spheres and for $n \geq 1$ let $\tilde{X}_n = \mathbf{R}^n \times S^1 \times S^1 \times \cdots$. Define $p_n\colon \tilde{X}_n \to X$ by